Lec 2

Compute probabilities for a Markov chain?

Need $P = \begin{bmatrix} & & \\ & p_{ij} & \\ & & \end{bmatrix}$ transition matrix (size $k \times k$ where k = size of state space)

and $\underline{\phi}_0 = [\; \cdots \;]$ ← initial probabilities

$\phi_0(i) = P(X_0 = i)$

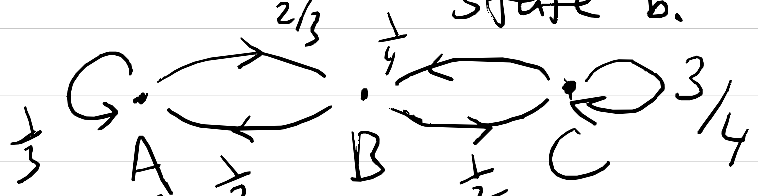
e.g. state space of size 4.

$\phi_0 = [\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}]$ means we start at any state equally likely.

states → a b c d

$\phi_0 = [0\ 1\ 0\ 0]$ means we start at state b.

E.g.



$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix} \end{array}$$

Note: rows of P add up to 1

$\left(\sum_{j \in S} P(X_n = j \mid X_{n-1} = i) = 1 \right)$ ← state space

P is a stochastic matrix
(entries ≥ 0 and rows add up to 1)

Also, entries of Φ_0 add up to 1:

$$\left(\sum_{i \in S} P(X_0 = i) = 1 \right)$$

How to calculate $P(\cancel{X}_1 = j)$?

Law of total probability:

$$P(X_1 = j) = \sum_{i \in S} P(X_1 = j, X_0 = i)$$

$$= \sum_{i \in S} P(X_0 = i) \, P(X_1 = j \mid X_0 = i)$$

$$= \sum_{i \in S} \Phi_0(i) \, P_{ij} = j^{th} \text{ entry of } \Phi_0 P$$

So if $\Phi_1 = [\ \cdots\ ]$ ← w/ entries $\Phi_1(i) = P(X_1 = i)$

then $\boxed{\Phi_1 = \Phi_0 P}$

Looks like linear algebra may help!

Time-homogenous Markov chain:

$\phi_n = [\cdots]$ w/ entries $\phi_n(i) = P(X_n = i)$

$$\underline{\phi}_n = \underline{\phi}_{n-1} P$$

so $\boxed{\underline{\phi}_n = \underline{\phi}_0 P^n}$

i^{th} entry

Special case: $\underline{\phi}_0 = [0 \cdots 0\ 1\ 0 \cdots 0]$
(i.e. start at state i)

$\rightarrow P(X_n = j \mid X_0 = i) = \underline{\phi}_n(j) = j^{th}$ entry of $\underline{\phi}_0 P^n$

i^{th} row of P^n

Thus: $P(X_n = j \mid X_0 = i)$ is the (i,j) entry of P^n

n-step transition probab. from i to j

time homogenous $\Rightarrow$

$$P(X_4 = j \mid X_2 = i) = P(X_2 = j \mid X_0 = i) = (P^2)_{ij}$$

$$P(X_4 = j, X_7 = k \mid X_0 = i) = (P^2)_{ij} \cdot (P^3)_{jk}$$

$$P(X_2 = i, X_4 = j) = \underbrace{P(X_2 = i)}_{(\phi_0 P^2)_i} (P^2)_{ij}$$

while:

$$P(X_0 = i, X_2 = j) = \underbrace{P(X_0 = i)}_{\phi_0(i)} (P^2)_{ij}$$

different

What about: $P(X_0 = i \mid X_1 = j)$?

Bayes: $P(X_0 = i \mid X_1 = j) = \dfrac{P(X_0 = i, X_1 = j)}{P(X_1 = j)} = \dfrac{P(X_0 = i)\, P(X_1 = j \mid X_0 = i)}{P(X_1 = j)}$