MATH 5040/6810: HOMEWORK 4 (DUE MONDAY, NOVEMBER 17)

Problem 1. Consider the following branching process X_n. Generation 0 has 1 individual, i.e. $X_0 = 1$. Then, each individual of each generation, independently of all other individuals, gives at most 2 descendants and immediately dies. Let $p \in [0, 1]$ be the probability an individual does not give any descendants and let $q \in [0, 1]$ be the probability the individual gives 2 descendants. Of course, $p + q \le 1$ (why?).

(a) What is the condition on p and q for this process to have a chance at survival?
(b) Compute the probability of survival. (Hint: probability-generating function.)

(a) We have $p_0 = p$, $p_2 = q$, and thus $p_1 = 1 - p - q$ (and this is why $p + q \le 1$). Then the mean number of offsprings is $0 \times p + 1 \times (1 - p - q) + 2 \times q = 1 - p + q$. To have a chance at survival we must have $\mu > 1$ which means $q > p$.

(b) We need to solve $\Phi(\alpha) = \alpha$. Here $\Phi(\alpha) = \alpha^0 \times p + \alpha^1 \times (1 - p - q) + \alpha^2 \times q = q\alpha^2 + (1 - p - q)\alpha + p$ and the equation is

$$q\alpha^2 + (1 - p - q)\alpha + p = \alpha$$

or equivalently

$$q\alpha^2 - (p + q)\alpha + p = 0.$$

We know $\alpha = 1$ is always a solution (and we can see that here by just plugging it in). So we factor the above quadratic polynomial as $(\alpha - 1)(q\alpha - p)$ to get that the other solution is $\alpha = p/q$. This is the probability of survival. (Note how for this to be a number less than 1 we need $q > p$.)

Problem 2. Let T and S be two independent exponential random variables. Let λ be the rate of T and μ the rate of S.

(a) What is the distribution of the first order statistic, i.e. random variable $\min(T, S)$?
(b) Calculate the probability of $T < S$.

Compute

$$P\{\min(T, S) > a) = P(T > a, S > a) = P(T > a)P(S > a) = e^{-\lambda a}e^{-\mu a} = e^{-(\lambda+\mu)a}.$$

We used the independence of T and S for the second equality. The above shows that $\min(T, S)$ has the CDF of an exponential random variable with rate $\lambda + \mu$. Also,

$$P(T < S) = \lambda\mu \iint_{0<t<s} e^{-\lambda t}e^{-\mu s}\,ds\,dt = \lambda\mu \int_0^\infty e^{-\lambda t}\Big(\int_t^\infty e^{-\mu s}\,ds\Big)\,dt = \lambda \int_0^\infty e^{-\lambda t}e^{-\mu t}\,dt = \frac{\lambda}{\lambda + \mu}.$$

Problem 3. Consider a continuous-time Markov process X_t on the finite state-space $\{1, 2, 3\}$. Let $a(1, 2)$, $a(1, 3)$, $a(2, 1)$, $a(2, 3)$, $a(3, 1)$, and $a(3, 2)$ be the rates of the Markov chain's exponential clocks. Let $a(1) = a(1, 2) + a(1, 3)$. Define $a(2)$ and $a(3)$ similarly.

Let Y_t be a stochastic process that does the following: if currently $Y_t = 1$, then the process waits for an exponential clock with rate $a(1)$ to ring. Once it rings, the process jumps to 2 or 3 with probabilities $a(1, 2)/a(1)$ and $a(1, 3)/a(1)$, respectively. The behavior is similar if Y_t is at 2 or at 3. Clearly, this defines a continuous-time Markov process.

(a) Let $X_0 = 1$ and fix time $t > 0$. Compute the probability that process X will make at least one jump by time t. (Hint: to make a jump one of the two clocks on arrows $1 \to 2$ and $1 \to 3$ has to ring. Now use Problem 2(a) to find the distribution of the time of the first jump.)

(b) Let $Y_0 = 1$ and fix time $t > 0$. Compute the probability that process Y will make at least one jump by time t. (Hint: to make a jump the clock at 1 needs to ring.)

(c) If you did not get the same answer in (a) and (b) you did something wrong. Go back and fix it.

(d) Let $X_0 = 1$. Compute the probability that when process X jumps, it jumps to state 2. (Hint: this means $1 \to 2$ rang before $1 \to 3$. Now use Problem 2(b).)

(e) Let $Y_0 = 1$. Compute the probability that when process Y jumps, it jumps to state 2. (Hint: read the definition of process Y carefully!)

(f) If you did not get the same answer in (d) and (e) you did something wrong. Go back and fix it.

Remark: Of course, in the above the choice of 1 and 2 as states is immaterial. The result holds for any two states, thus showing that statistically process Y makes the same moves as process X. In other words, to simulate process X one can instead simulate process Y. The advantage is that process Y uses only exponential clocks placed on the states, rather than on the arrows from state to state (which is how process X works). So process Y uses fewer exponential clocks, and thus is easier to simulate.

(a) The time of the first jump of X out of 1 is the minimum of the times for the jump from 1 to 2 and from 1 to 3. By Problem 2(a) this is an exponential random variable with rate $a(1,2) + a(1,3) = a(1)$. Hence, the probability of at least one jump by time t, which is the probability the time of the first jump is less than t, equals $1 - e^{-a(1)t}$ (the CDF of the exponential with rate $a(1)$).

(b&c) The time of the first jump of Y out of 1 is an exponential with rate $a(1)$ and so the probability the first jump happens before time t is its CDF, which gives the same answer as for (a).

(d) The probability X jumps from 1 to 2 is exactly the probability the clock from 1 to 2 rings before the clock from 1 to 3. By Problem 2(b), the probability is proportional to the rates of the two clocks and thus equals $a(1,2)/(a(1,2) + a(1,3)) = a(1,2)/a(1)$.

(e&f) The probability is by definition $a(1,2)/a(1)$, which is the same as in part (d).

Problem 4. Consider the same continuous-time Markov process X_t as in Problem 3. Let $\lambda = \max(a(1), a(2), a(3))$.

Let Z_t be a stochastic process that does the following: where Z_t is at the moment, it waits for an exponential clock with rate λ to ring. Say currently $Z_t = 1$ and the clock rang. Then the process jumps to 2 or 3 with probabilities $a(1,2)/\lambda$ and $a(1,3)/\lambda$, respectively. It stays at 1 with probability $1 - a(1)/\lambda$. The behavior is similar if Z_t is were at 2 or at 3 when the clock rang. Clearly, this defines a continuous-time Markov process.

(a) Let $Z_0 = 1$ and fix time $t > 0$. Compute the probability that process Z will make at least one jump by time t. Jumping to 1 does not count as a jump. (Hint: to make a jump the clock needs to ring AND the process needs to not stay put. Recall now the computation we did in class when a clock rings but you do not always listen to it, then this is equivalent to another exponential clock with the rate being multiplied by the probability you do react to the clock.) You should get the same answer as in (a) and (b) of Problem 3.

(b) Let $Z_0 = 1$. Compute the probability that when process Z actually jumps, it jumps to state 2. (Hint: this is a Math 5010 problem with conditional probabilities.)

Remark: Again, the above shows that statistically process Z makes the same moves as process X. In other words, to simulate process X one can instead simulate process Z. The advantage over using process Y is that process Z uses only ONE exponential clock. Hence, it is much simpler to simulate. Note that with process Z the clock rings more often (since it has a rate larger than all the rates of clocks in processes X and Y.) However, process Z has the option of staying put. At the end of the day, although the clock rings more often, the process still behaves like X.

Remark: Now, look back at how the above two homework problems worked out. Notice that there was nothing special about the size of the state space, as long as it is finite. In other words, what these problems achieve is true for any finite state space. We just worked it out for 3 states to make it easier to conceive.

(a) Since the probability for Z to jump out of 1 when the clock rings is equal to $a(1)/\lambda$ (by definition of the Z process), the hint says that the time of the first jump of Z out of 1 is an exponential random variable with rate $\lambda \times a(1)/\lambda = a(1)$. Thus, the probability the first jump happens before time t is $1 - e^{-a(1)t}$. This matches the previous results in Problem 3(a&b).

(b) This is asking for the probability to go from 1 to 2, given that the process does jump away from 1. The intersection of the two events is simply the event that the process jumps from 1 to 2 (probability of $a(1,2)/\lambda$). The probability that the process does jump away from 1 is $a(1)/\lambda$. The conditional probability we are after is then $a(1,2)/a(1)$.

Problem 5. Consider a continuous time Markov chain with states $1, 2, 3, 4$. From 1 it jumps to any of the other states at rate 1 each. From 2 it jumps to 3 at rate 2 and to 4 at rate 1. From 3 it jumps to 1 or 4 at rate 1 and to 2 at rate 2. From 4 it can only jump to 3, and does that at rate 1.

(a) Draw the graph for this chain and write down the rates matrix (also known as the infinitesimal generator of the Markov chain).
(b) Find the equilibrium distribution.
(c) Suppose the chain starts in state 1. What is the expected amount of time it requires to jump away from that state? Why?
(d) Assume the chain starts in state 2. What is the expected amount of time until the chain is in state 1?
(e) Turn this Markov chain into one that uses one exponential clock and a transition matrix of a discrete Markov chain. What is the rate of the common clock, and what is the transition matrix?
(f) Compute the invariant measure of the discrete Markov chain.

(g) Assume the discrete Markov chain starts in state 2. What is the expected amount of steps it will take until it is in state 1?

(h) Use the answer from (g) to recalculate the answer to (d).

(a) I leave the drawing out. The generator is

$$A = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

(b) The invariant measure is found by solving $\pi A = 0$. Letting $\pi = (x, y, z, u)$ we get the equations

$$-3x + z = 0, \quad x - 3y + 2z = 0, \quad x + y + z - u = 0, \quad \text{and} \quad x + y + z + u = 1.$$

(I omitted the equation coming from multiplying by the third column, since I know the four equations are redundant anyway and the right fourth equation is the one above.) Solving the above we get $z = 3x$, $3y = 7x$, $u = 4x + y$ and hence $3u = 19x$. Then $3x + 7x + 19x + 9x = 1$ and $x = 1/38$. From this we get $\pi = (3/38, 7/38, 9/38, 19/38)$.

(c) The time of first jump from 1 is an exponential random variable with rate 3. Its average is $1/3$.

(d) For this computation we need to omit the row and column of A corresponding to state 2, which gives

$$\widetilde{A} = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -4 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

We then compute

$$-\widetilde{A}^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{2}{3} & 1 & \frac{8}{3} \end{bmatrix}.$$

The time it takes to go from 2 to 1 is then the sum of the row corresponding to state 2, which is $1 + 1 + 2 = 4$ time units.

(e) The universal exponential clock would have a rate that is the maximum of all the rates corresponding to jumping out of the different states. These rates are $a(1) = a(2) = 3$, $a(3) = 4$, and $a(4) = 1$. Hence, the universal clock has rate 4. Once the clock rings, the Markov chain jumps like a discrete time chain with transition matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

(f) The invariant measure is found by solving $\pi P = \pi$. This gives the same answer as for part (b). (Check that, though, either by actually solving $\pi P = \pi$ or by plugging in the π from (b) and checking the equation is indeed satisfied.)

(g) We get the matrix Q by omitting the row and column of P corresponding to state 1 (which means we are making state 1 absorbing). We get

$$Q = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

Then

$$(I - Q)^{-1} = \begin{bmatrix} 4 & 4 & 8 \\ \frac{8}{3} & 4 & \frac{20}{3} \\ \frac{8}{3} & 4 & \frac{32}{3} \end{bmatrix}.$$

The average number of steps to go from 2 to 1 is the sum of the row entries corresponding to state 2, that is: $4 + 4 + 8 = 16$ steps.

(h) To recover the continuous-time Markov chain from the discrete-time one we simply think of the discrete-time chain jumping when the universal clock rings instead of at every time unit. Therefore, if the discrete-time chain makes n jumps, the continuous-time chain will be at that same position but at time $T_1 + \cdots + T_n$, where T_i are i.i.d. exponential random variables, independent of the dynamics of the discrete-time chain, and with rate equal to the universal rate we computed in (e), i.e. rate 4. Hence, the average time it takes the continuous-time chain to make n jumps is $n/4$. To go from 2 to 1 we found out in (g) that the chain needs on average 16 jumps. It will thus take $16/4 = 4$ time units. This is consistent with our finding in (d).

Note that we even have that $\frac{1}{4}(I - Q)^{-1} = -\widetilde{A}^{-1}$. This says that e.g. starting at 3 the continuous-time chain spends on average $5/3$ time units at state 4 before reaching state 1.