

MATH 5040/6810: HOMEWORK 4 (DUE MONDAY, NOVEMBER 17)

Problem 1. Consider the following branching process X_n . Generation 0 has 1 individual, i.e. $X_0 = 1$. Then, each individual of each generation, independently of all other individuals, gives at most 2 descendants and immediately dies. Let $p \in [0, 1]$ be the probability an individual does not give any descendants and let $q \in [0, 1]$ be the probability the individual gives 2 descendants. Of course, $p + q \leq 1$ (why?).

- (a) What is the condition on p and q for this process to have a chance at survival?
- (b) Compute the probability of survival.

Problem 2. Let T and S be two independent exponential random variables. Let λ be the rate of T and μ the rate of S .

- (a) What is the distribution of the first order statistic, i.e. random variable $\min(T, S)$?
- (b) Calculate the probability of $T < S$.

Problem 3. Consider a continuous-time Markov process X_t on the finite state-space $\{1, 2, 3\}$. Let $a(1, 2)$, $a(1, 3)$, $a(2, 1)$, $a(2, 3)$, $a(3, 1)$, and $a(3, 2)$ be the rates of the Markov chain's exponential clocks. Let $a(1) = a(1, 2) + a(1, 3)$. Define $a(2)$ and $a(3)$ similarly.

Let Y_t be a stochastic process that does the following: if currently $Y_t = 1$, then the process waits for an exponential clock with rate $a(1)$ to ring. Once it rings, the process jumps to 2 or 3 with probabilities $a(1, 2)/a(1)$ and $a(1, 3)/a(1)$, respectively. The behavior is similar if Y_t is at 2 or at 3. Clearly, this defines a continuous-time Markov process.

- (a) Let $X_0 = 1$ and fix time $t > 0$. Compute the probability that process X will make at least one jump by time t . (Hint: to make a jump one of the two clocks on arrows $1 \rightarrow 2$ and $1 \rightarrow 3$ has to ring. Now use Problem 2(a) to find the distribution of the time of the first jump.)
- (b) Let $Y_0 = 1$ and fix time $t > 0$. Compute the probability that process Y will make at least one jump by time t . (Hint: to make a jump the clock at 1 needs to ring.)
- (c) If you did not get the same answer in (a) and (b) you did something wrong. Go back and fix it.
- (d) Let $X_0 = 1$. Compute the probability that when process X jumps, it jumps to state 2. (Hint: this means $1 \rightarrow 2$ rang before $1 \rightarrow 3$. Now use Problem 2(b).)
- (e) Let $Y_0 = 1$. Compute the probability that when process Y jumps, it jumps to state 2. (Hint: read the definition of process Y carefully!)
- (f) If you did not get the same answer in (d) and (e) you did something wrong. Go back and fix it.

Remark: Of course, in the above the choice of 1 and 2 as states is immaterial. The result holds for any two states, thus showing that statistically process Y makes the same moves as process X . In other words, to simulate process X one can instead simulate process Y . The advantage is that process Y uses only exponential clocks placed on the states, rather than on the arrows from state to state (which is how process X works). So process Y uses fewer exponential clocks, and thus is easier to simulate.

Problem 4. Consider the same continuous-time Markov process X_t as in Problem 3. Let $\lambda = \max(a(1), a(2), a(3))$.

Let Z_t be a stochastic process that does the following: where Z_t is at the moment, it waits for an exponential clock with rate λ to ring. Say currently $Z_t = 1$ and the clock rang. Then the process jumps to 2 or 3 with probabilities $a(1, 2)/\lambda$ and $a(1, 3)/\lambda$, respectively. It stays at 1 with probability $1 - a(1)/\lambda$. The behavior is similar if Z_t is were at 2 or at 3 when the clock rang. Clearly, this defines a continuous-time Markov process.

- (a) Let $Z_0 = 1$ and fix time $t > 0$. Compute the probability that process Z will make at least one jump by time t . Jumping to 1 does not count as a jump. (Hint: to make a jump the clock needs to ring AND the process needs to not stay put. Recall now the computation we did in class when a clock rings but you do not always listen to it, then this is equivalent to another exponential clock with the rate being multiplied by the probability you do react to the clock.) You should get the same answer as in (a) and (b) of Problem 3.
- (b) Let $Z_0 = 1$. Compute the probability that when process Z actually jumps, it jumps to state 2. (Hint: this is a Math 5010 problem with conditional probabilities.)

Remark: Again, the above shows that statistically process Z makes the same moves as process X . In other words, to simulate process X one can instead simulate process Z . The advantage over using process Y is that process Z uses only ONE exponential clock. Hence, it is much simpler to simulate. Note that with process Z the clock rings more often (since it has a rate larger than all the rates of clocks in processes X and Y .) However, process Z has the option of staying put. At the end of the day, although the clock rings more often, the process still behaves like X .

Remark: Now, look back at how the above two homework problems worked out. Notice that there was nothing special about the size of the state space, as long as it is finite. In other words, what these problems achieve is true for any finite state space. We just worked it out for 3 states to make it easier to conceive.

Problem 5. Consider a continuous time Markov chain with states 1, 2, 3, 4. From 1 it jumps to any of the other states at rate 1 each. From 2 it jumps to 3 at rate 2 and to 4 at rate 1. From 3 it jumps to 1 or 4 at rate 1 and to 2 at rate 2. From 4 it can only jump to 3, and does that at rate 1.

- (a) Draw the graph for this chain and write down the rates matrix (also known as the infinitesimal generator of the Markov chain).
- (b) Find the invariant measure.
- (c) Suppose the chain starts in state 1. What is the expected amount of time it requires to jump away from that state?
- (d) Assume the chain starts in state 2. What is the expected amount of time until the chain is in state 1?
- (e) Turn this Markov chain into one that uses one exponential clock and a transition matrix of a discrete Markov chain. What is the rate of the common clock, and what is the transition matrix?
- (f) Compute the invariant measure of the discrete Markov chain.
- (g) Assume the discrete Markov chain starts in state 2. What is the expected amount of steps it will take until it is in state 1?
- (h) Use the answer from (g) to recalculate the answer to (d).