Homework 3 Problem 4, continued

Suppose we are given a probability mass function \( p \) on the nonnegative integers. That is, \( p(x) \geq 0 \) for all \( x = 0, 1, 2, \ldots \), and \( \sum_{x=0}^{\infty} p(x) = 1 \). Consider the (“renewal”) Markov chain on nonnegative integers that does the following: from 0 it jumps to site \( x \geq 0 \) with probability \( p(x) \) and from a site \( x \geq 1 \) it goes to \( x - 1 \) with probability 1. In other words, the chain moves to the left until it hits 0, and from 0 it picks a site at random (according to the probability mass function \( p \)) to jump to, then moves to the left again, and so on. In Problem 4 on Homework 3 you determined when the chain is transient and when it is recurrent. Next, figure out whether or not the chain has an invariant measure. What does it mean when the chain does have an invariant measure?

Solution: Since the only sites that can go to \( x \) are \( x + 1 \) (which goes to \( x \) with probability 1) and 0 (which goes to \( x \) with probability \( p(x) \)), the equation for the invariant measure \( \pi \) is

\[
\pi(x) = \pi(x + 1) + p(x)\pi(0).
\]

Rearranging and using \( x - 1 \) in place of \( x \) we have

\[
\pi(x) - \pi(x - 1) = -p(x - 1)\pi(0) \quad \text{for all } x \geq 1.
\]

Adding these up and cancelling things in the telescoping sum on the left we get

\[
\pi(x) - \pi(0) = -\pi(0)(p(0) + p(1) + \ldots + p(x - 1)).
\]

So

\[
\pi(x) = \pi(0)(1 - p(0) - p(1) - \ldots - p(x - 1)) \quad \text{for all } x \geq 1.
\]

Note that since \( p \) adds up to 1 we have

\[
1 - p(0) - p(1) - \ldots - p(x - 1) = \sum_{y=x}^{\infty} p(y).
\]

So

\[
\pi(x) = \pi(0) \sum_{y=x}^{\infty} p(y) \quad \text{for all } x \geq 1.
\]

This also happens to hold for \( x = 0 \) (since it just gives \( \pi(0) = \pi(0) \)). So this formula holds in fact for all \( x \geq 0 \).

To determine \( \pi(0) \) we use the fact that \( \pi \) needs to add up to 1 and write

\[
1 = \sum_{x=0}^{\infty} \pi(x) = \pi(0) \sum_{x=0}^{\infty} \sum_{y=x}^{\infty} p(y) = \pi(0) \sum_{y=0}^{\infty} \sum_{x=0}^{y} p(y) = \pi(0) \sum_{y=0}^{\infty} \sum_{x=0}^{(y+1)} p(y) = \pi(0) \left( 1 + \sum_{y=0}^{\infty} yp(y) \right).
\]

So an invariant measure exists if and only if \( \sum_{y=0}^{\infty} yp(y) < \infty \), which matches the finding that the chain is positive recurrent if an only if this sum is finite. Furthermore, we get that \( 1 = \pi(0)E[T_0] \), which matches the formula that says \( E[T_0] = 1/\pi(0) \). Finally, the formula for the invariant measure is

\[
\pi(x) = \frac{\sum_{y=x}^{\infty} p(y)}{1 + \sum_{y=0}^{\infty} yp(y)}.
\]
Problem 6

Consider the Markov chain with state space \( \{0, 1, 2, \ldots\} \) and transition probability
\[
p(x, x + 1) = \frac{x + 1}{2x + 1}, \quad p(x, x - 1) = \frac{1}{2}, \quad p(x, x) = 1 - p(x, x + 1) - p(x, x - 1), \quad \text{for } x \geq 1,
\]
p(0, 0) = \frac{3}{4}, and p(0, 1) = \frac{1}{4}. Find its invariant measure. Is it transient, null recurrent, or positive recurrent?

**Hint:** In class, we worked out formulas for \( \alpha(x) = P(\text{chain ever visits 0}|X_0 = x) \) and for the invariant measure \( \pi \) (if it exists), for a Markov chain on \( \{0, 1, 2, \ldots\} \) with nearest-neighbor transition probabilities that can vary with the location. The Markov chain here is a special case, with a specific choice of these transition probabilities. So apply the appropriate formulas from the lecture (no need to rederive them).

**Solution:** This is a birth-and-death process that we studied in class. The general form of this was a process that has state space \( 0, 1, 2, \ldots \), and from \( x \geq 1 \) the process jumps to \( x + 1 \) with probability \( a_x \), to \( x - 1 \) with probability \( b_x \), and stays at \( x \) with probability \( c_x \).

From \( x = 0 \) the process simply does not jump back and so \( b_0 = 0 \). Then, we found in class that the invariant measure has the formula
\[
\pi(x) = \pi(0) \frac{a_0 \cdots a_{x-1}}{b_1 \cdots b_x} \quad \text{for all } x \geq 1.
\]

In this exercise we have \( a_k = \frac{k+1}{2(k+2)} \) and \( b_k = \frac{1}{2} \). So
\[
a_0 \cdots a_{x-1} = \frac{1}{2(2)} \cdot \frac{2}{2(3)} \cdot \frac{3}{2(4)} \cdot \frac{4}{2(5)} \cdots \frac{x}{2(x+1)} = \frac{1}{2x(x+1)}
\]
and
\[
b_1 \cdots b_x = \frac{1}{2^x}.
\]
Taking the ratio gives
\[
\pi(x) = \frac{\pi(0)}{x+1} \quad \text{for all } x \geq 1.
\]
The formula holds also for \( x = 0 \) because it gives \( \pi(0) = \pi(0) \). To determine \( \pi(0) \) we use the fact that \( \pi \) needs to add up to 1. But \( \sum_{x=0}^{\infty} \frac{1}{x+1} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty \) and so there is no way to choose \( \pi(0) \) to make the above add up to 1. Hence, there is no invariant measure for this Markov chain. This means it is not positive recurrent. But it could still be null recurrent or transient.

To determine whether the chain is transient or not we need to compute the \( \alpha \). In class, we found that \( \alpha \) is given by \( \alpha(0) = 1 \), \( \alpha(1) \) is to be determined, and
\[
\alpha(x) = 1 - (1 - \alpha(1)) \left( 1 + \sum_{k=1}^{x-1} \frac{b_k}{a_1 \cdots a_k} \right) \quad \text{for } x \geq 2.
\]
In our example, we have \( b_1 \cdots b_k = \frac{1}{2^k} \) and \( a_1 \cdots a_k = \frac{2}{2(3)} \cdot \frac{3}{2(4)} \cdot \frac{4}{2(5)} \cdots \frac{k+1}{2(k+2)} = \frac{2}{2^k(k+2)} \). Therefore,
\[
\alpha(x) = 1 - (1 - \alpha(1)) \left( 1 + \sum_{k=1}^{x-1} \frac{k+2}{2} \right).
\]
The sum clearly goes to infinity as \( x \to \infty \). Therefore, to have \( \alpha \) remain bounded between 0 and 1 we have to take \( \alpha(1) = 1 \). But then we get \( \alpha(x) = 1 \) for all \( x \), which still does not
add up to 1. Hence, there is no $\alpha$ that satisfies the necessary assumptions for transience and we have recurrence. Since we dismissed positive recurrence, the chain is null recurrent, just like the simple symmetric one-dimensional random walk.

*Remark* 0.1. Note how if we instead defined $p(x, x + 1) = \frac{(x + 1)^2}{2(x + 2)^2}$, then the same computations as above would have given that $\pi(x) = \frac{\pi(0)}{(x + 1)^2}$. But now, since $\sum_{x=0}^{\infty} \frac{1}{(x + 1)^2} < \infty$, we do have an invariant measure given by

$$
\pi(x) = \sum_{k=1}^{\infty} \frac{1}{x + 1} \quad \text{for} \ x \geq 0.
$$

In this case, the Markov chain would be positive recurrent, in contrast to the simple symmetric random walk.

**Problem 7**

Consider the Markov chain with state space $\{0, 1, 2, \ldots\}$ and transition probability $p(x, x - 1) = \frac{1}{2}(1 - \frac{1}{x+2})$ and $p(x, x + 1) = \frac{1}{2}(1 + \frac{1}{x+2})$ for $x \geq 1$, $p(0, 1) = \frac{3}{4}$, and $p(0, 0) = \frac{1}{4}$. Show that this chain is transient.

**Hint:** Same hint as the above question.

**Remark:** In Problems 6 and 7, when $x$ gets large both Markov chains have transition probabilities that are close to the ones for the simple symmetric random walk. However, note how a small perturbation of the transition probabilities is causing them to behave very differently from the simple symmetric random walk (which we know is null recurrent)!

**Solution:** This is again a birth-and-death process. We can compute the $\alpha$ with the same formula as above. But here, $b_k = \frac{1}{2}(1 - \frac{1}{k+2}) = \frac{k+1}{2(k+2)}$, which was the formula for $a_k$ in the previous question. So now $b_1 \cdots b_k = \frac{2}{2^{k+2}}$. But in this example $a_k = \frac{1}{2}(1 + \frac{1}{k+2}) = \frac{k+3}{2(k+2)}$. Therefore, $a_1 \cdots a_k = \frac{4}{2^3} \cdot \frac{5}{2^4} \cdots \frac{k+3}{2^{k+1}} = \frac{k+3}{2^{k+3}}$. Thus,

$$
\alpha(x) = 1 - (1 - \alpha(1)) \left(1 + \sum_{k=1}^{x-1} \frac{6}{(k + 2)(k + 3)} \right).
$$

In contrast to the previous example, here we have

$$
\sum_{k=1}^{x-1} \frac{6}{(k + 2)(k + 3)} = 6 \sum_{k=1}^{x-1} \left( \frac{1}{k + 2} - \frac{1}{k + 3} \right) = 6 \left( \frac{1}{3} - \frac{1}{x + 2} \right) = \frac{2(x - 1)}{x + 2}.
$$

But then the formula for $\alpha$ simplifies to

$$
\alpha(x) = 1 - (1 - \alpha(1)) \left(1 + \frac{2(x - 1)}{x + 2} \right) = 1 - (1 - \alpha(1)) \frac{3x}{x + 2}.
$$

Since we want $\alpha(x) \to 0$ as $x \to \infty$ for transience to happen, this would mean that

$$
0 = 1 - 3(1 - \alpha(1)).
$$
Solving, we get $\alpha(1) = \frac{2}{3}$ and

$$\alpha(x) = 1 - \frac{x}{x + 2} = \frac{2}{x + 2} \quad \text{for} \ x \geq 1.$$  

The formula also happens to work for $x = 0$ because it gives $\alpha(0) = 1$, as desired. Since we found an $\alpha$ that solves the appropriate equation and satisfies the necessary boundary conditions for transience, we have shown the Markov chain is transient.

**Problem 8**

Suppose we are given numbers $p(x) \in (0, 1)$ for all integers $x \geq 0$. Consider the “aging chain” on \{0, 1, 2, \ldots\} in which from each $x \geq 0$ the chain moves from $x$ to $x + 1$ (gets “older” by 1) with probability $p(x)$ and moves back to 0 (“dies”) with probability $1 - p(x)$.

(a) Is this chain irreducible or not?

(b) What conditions on $p$ give transience? **Hint:** Starting at 0, what needs to happen to never return to 0?

(c) What conditions on $p$ give positive recurrence? **Hint:** Calculate the invariant measure.

**Solution:** (a) The chain is irreducible because for any $x$ and $y$, if $x < y$ then the chain can just march from $x$ to $y$ and if $y < x$, then then chain can jump from $x$ to 0 and then march to $y$. So $x$ and $y$ communicate.

(b) Starting at 0, the chain never comes back to 0 only if it keeps marching to the right without ever jumping back to 0. This happens with probability $p(0)p(1)p(2)\cdots$. So the chain is transient if and only if this infinite product is positive and it is recurrent if the infinite product is 0.

(c) The invariant measure satisfies $\pi(x) = p(x - 1)\pi(x - 1)$ for all $x \geq 1$. Therefore,

$$\pi(x) = p(x - 1)p(x - 2)\cdots p(0)\pi(0).$$

The chain is thus positive recurrent if

$$\sum_{x=1}^{\infty} p(x - 1)p(x - 2)\cdots p(0) < \infty$$

and it is not positive recurrent otherwise.