

Problem 7

Consider the Markov chain on nonnegative integers that does the following: from 0 it jumps to site $x \geq 0$ with probability $p(x)$ [of course, $p(x) \geq 0$ for all $x \geq 0$ and $\sum_{x=0}^{\infty} p(x) = 1$]. From a site $x > 0$ it goes to $x - 1$ with probability 1. In other words, the chain moves to the left until it hits 0, and from 0 it picks a site at random to jump to, then moves to the left again, and so on.

- (a) Is this chain irreducible? If not, what are the communicating classes? Is it aperiodic? If not, what are the periods of the different classes?
- (b) Starting at state 0, calculate the probability distribution of T_0 , the time of first return to 0. What is $E[T_0]$?
- (c) Is this chain transient, null recurrent, or positive recurrent? Justify your answer.
- (d) Calculate the invariant measure, whenever it exists. What does it mean when the chain does have an invariant measure?

Problem 8

Consider the Markov chain with state space $\{0, 1, 2, \dots\}$ and transition probability $p(x, x+1) = \frac{x+1}{2x+4}$, $p(x, x-1) = \frac{1}{2}$, and $p(x, x) = 1 - p(x, x+1) - p(x, x-1)$, for $x \geq 1$, $p(0, 0) = \frac{3}{4}$, and $p(0, 1) = \frac{1}{4}$. Find its invariant measure. Is it transient, null recurrent, or positive recurrent? **Hint:** This is a birth-and-death process.

Remark: In Problems 5 and 8, when x gets large both Markov chains have transition probabilities that are close to the ones for the simple symmetric random walk. However, note how a small perturbation of the transition probabilities is causing them to behave very differently from the simple symmetric random walk (which we know is null recurrent)!

Problem 9

Suppose we are given numbers $p(x) \in (0, 1)$ for all integers $x \geq 0$. Consider the “aging chain” on $\{0, 1, 2, \dots\}$ in which from each $x \geq 0$ the chain moves from x to $x+1$ (gets “older” by 1) with probability $p(x)$ and moves back to 0 (“dies”) with probability $1 - p(x)$.

- (a) Is this chain irreducible or not?
- (b) What conditions on p give transience? **Hint:** Starting at 0, what needs to happen to never return to 0?
- (c) What conditions on p give positive recurrence? **Hint:** Calculate the invariant measure.