Homework 3 Problem 4, continued

Suppose we are given a probability mass function \( p \) on the nonnegative integers. That is, \( p(x) \geq 0 \) for all \( x = 0, 1, 2, \ldots \), and \( \sum_{x=0}^{\infty} p(x) = 1 \). Consider the (“renewal”) Markov chain on nonnegative integers that does the following: from 0 it jumps to site \( x \geq 0 \) with probability \( p(x) \) and from a site \( x \geq 1 \) it goes to \( x - 1 \) with probability 1. In other words, the chain moves to the left until it hits 0, and from 0 it picks a site at random (according to the probability mass function \( p \)) to jump to, then moves to the left again, and so on.

In Problem 4 on Homework 3 you determined when the chain is transient and when it is recurrent. Next, figure out whether or not the chain has an invariant measure. What does it mean when the chain does have an invariant measure?

Problem 6

Consider the Markov chain with state space \( \{0, 1, 2, \ldots \} \) and transition probability \( p(x, x + 1) = \frac{x+1}{2x+4} \), \( p(x, x - 1) = \frac{1}{2(1-x+2)} \) for \( x \geq 1 \), \( p(0, 0) = \frac{3}{4} \), and \( p(0, 1) = \frac{1}{4} \). Find its invariant measure. Is it transient, null recurrent, or positive recurrent?

Hint: In class, we worked out formulas for \( \alpha(x) = P(\text{chain ever visits 0}|X_0 = x) \) and for the invariant measure \( \pi \) (if it exists), for a Markov chain on \( \{0, 1, 2, \ldots \} \) with nearest-neighbor transition probabilities that can vary with the location. The Markov chain here is a special case, with a specific choice of these transition probabilities. So apply the appropriate formulas from the lecture (no need to rederive them).

Problem 7

Consider the Markov chain with state space \( \{0, 1, 2, \ldots \} \) and transition probability \( p(x, x + 1) = \frac{1}{2}(1 + \frac{1}{x+2}) \) and \( p(x, x - 1) = \frac{1}{2}(1 - \frac{1}{x+2}) \) for \( x \geq 1 \), \( p(0, 1) = \frac{3}{4} \), and \( p(0, 0) = \frac{1}{4} \). Show that this chain is transient.

Hint: Same hint as the above question.

Remark: In Problems 6 and 7, when \( x \) gets large both Markov chains have transition probabilities that are close to the ones for the simple symmetric random walk. However, note how a small perturbation of the transition probabilities is causing them to behave very differently from the simple symmetric random walk (which we know is null recurrent)!

Problem 8

Suppose we are given numbers \( p(x) \in (0, 1) \) for all integers \( x \geq 0 \). Consider the “aging chain” on \( \{0, 1, 2, \ldots \} \) in which from each \( x \geq 0 \) the chain moves from \( x \) to \( x + 1 \) (gets “older” by 1) with probability \( p(x) \) and moves back to 0 (“dies”) with probability \( 1 - p(x) \).

(a) Is this chain irreducible or not?

(b) What conditions on \( p \) give transience? Hint: Starting at 0, what needs to happen to never return to 0?

(c) What conditions on \( p \) give positive recurrence? Hint: Calculate the invariant measure.