Consider the following continuous-time Markov process on state-space \{1, 2, 3, 4, 5, 6, 7\}: the process goes from 1 to 2 at rate 1, from 1 to 3 at rate 2, from 2 to 1 at rate 1, from 2 to 3 at rate 1, from 3 to 1 at rate 2, from 3 to 2 at rate 3, from 4 to 5 at rate 2, from 5 to 4 at rate 1, from 6 to 1 at rate 2, from 6 to 2 at rate 1, from 6 to 4 at rate 1, from 7 to 2 at rate 1, from 7 to 3 at rate 3, from 7 to 5 at rate 1, and from 7 to 6 at rate 1.

(a) What are the transient states of this process? What are the recurrent communication classes?

(b) Calculate the average amount of time it takes to get to 3, starting at the various states 1 through 7.

(c) Estimate the probability that the process is at 5 at time 10000, given it started at 4 then given it started at 5.

(d) Estimate the probability that the process is at 5 at time 10000, given it started at 1 then given it started at 2 then given it started at 3.

(e) How would you modify the process to get statistically the same Markov process, but using only one exponential clock?

(f) What is the probability the process reaches a site in the set \{4, 5\}, given it started at 6 and then given it started at 7?

(g) Estimate the probability that the process is at state 5 at time 10000, given it started at 6 then given it started at 7.
2. Consider the following continuous-time Markov process on the non-negative integers: from 0 you go to 1 with rate 1. From $n \geq 1$ you go to $n - 1$ with rate 1 and to $n + 1$ with rate $\frac{1}{n+1}$.

(a) Show that this process is positive recurrent.

(b) Give a formula for the invariant measure.
Extra Credit (No Partial Credit):
Consider the following continuous-time Markov process. From 0 the process jumps to site $n \geq 1$ with rate $\frac{1}{2n}$. From a site $n \geq 1$ the process jumps to $n - 1$ with rate 1. Calculate the invariant measure for this process. Hint: see if you can first transform the problem to a discrete-time Markov chain question.
(1a) \(\{1, 2, 3\}\) all communicate with each other and do not communicate with other sites. Thus, these form one recurrent communication class. The same holds for \(\{4, 5\}\). 6 and 7 are transient since the process can get from either of the two to the recurrent classes.

(1b) Since we cannot get to 3 starting at 4 or 5 and since there is a positive probability we never get to 3 if we start at 6 or 7, the expected time to reach 3 is infinite for all these starting points. Thus, we can now focus on the recurrent class \(\{1, 2, 3\}\). For this class, the rates matrix is given by

\[
A = \begin{bmatrix}
-3 & 1 & 2 \\
1 & -2 & 1 \\
2 & 3 & -5
\end{bmatrix}.
\]

Since we are asked about reaching site 3, we ignore its row and column and get to the matrix

\[
\tilde{A} = \begin{bmatrix}
-3 & 1 \\
1 & -2
\end{bmatrix}.
\]

Then

\[
(-\tilde{A})^{-1} = \begin{bmatrix}
3 & -1 \\
-1 & 2
\end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix}
2 & 1 \\
1 & 3
\end{bmatrix}.
\]

So, starting at 1 it takes, on average, \(\frac{2+1}{5} = \frac{3}{5}\) time units to get to 3. Starting at 2, it takes, on average, \(\frac{1+3}{5} = \frac{4}{5}\) time units to get to 3. Of course, starting at 3 it takes no time (i.e. 0 time units) to get to 3.

(1c) Since \(\{4, 5\}\) is a recurrent communication class, the question is answered by computing its invariant measure. The rates matrix for this class is given by

\[
A = \begin{bmatrix}
-2 & 2 \\
1 & -1
\end{bmatrix}.
\]

Suppose the invariant measure for this class is given by the vector \([xy]\). Then

\[
[x \ y] \begin{bmatrix}
-2 & 2 \\
1 & -1
\end{bmatrix} = [00],
\]

which gives \(-2x + y = 0\). We also know that we should have \(x + y = 1\). This leads to \(x = 1/3\) and \(y = 2/3\). Thus, starting at either 4 or 5, the probability the process is at 5 after a long time is \(2/3\).
(1d) Starting at any of \{1,2,3\}, we have no chance to get to 5. So the probability is 0.

(1e) The rates to move out of the sites 1 through 7 are, respectively, 3, 2, 5, 2, 1, 4, and 6. Therefore, for any \(x \neq y\) in \{1, \ldots, 7\} we would divide the rate of going from \(x\) to \(y\) by by 6 to get the transition probability from \(x\) to \(y\) for the discrete-time version of the process. Then, for \(x\) in \{1, \ldots, 7\}, the probability to go from \(x\) to \(x\) is computed to be one minus the probabilities to go from \(x\) to all the other sites (i.e. we populate the diagonal of the transition matrix by ensuring that its rows add up to one). Since the original rates matrix is

\[
A = \begin{bmatrix}
-3 & 1 & 2 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 \\
2 & 3 & -5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
2 & 1 & 0 & 1 & 0 & -4 & 0 \\
0 & 1 & 0 & 3 & 1 & 1 & -6
\end{bmatrix},
\]

The transition matrix for the discrete version is

\[
P = \begin{bmatrix}
3/6 & 1/6 & 2/6 & 0 & 0 & 0 & 0 \\
1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 \\
1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4/6 & 2/6 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/6 & 5/6 & 0 \\
2/6 & 1/6 & 0 & 1/6 & 0 & 6/6 & 0 \\
0 & 1/6 & 0 & 3/6 & 1/6 & 1/6 & 0
\end{bmatrix}.
\]

The continuous-time process would then follow the discrete-time Markov chain with the above transition matrix, but would make its moves following an exponential clock with rate 6.
(1f) For this we use the matrix $P$, after collapsing the recurrent classes into absorbing sites. This leads to the new matrix

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\frac{3}{6} & \frac{1}{6} & \frac{2}{6} & 0 \\
\frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0
\end{bmatrix}.
$$

The transient-to-transient matrix is

$$
Q = \begin{bmatrix}
\frac{2}{6} & 0 \\
\frac{1}{6} & 0
\end{bmatrix}.
$$

The transient-to-absorbing matrix is

$$
S = \begin{bmatrix}
\frac{3}{6} & \frac{1}{3} \\
\frac{1}{6} & \frac{2}{3}
\end{bmatrix}.
$$

To calculate the desired probability we compute

$$
M = (I - Q)^{-1} = \begin{bmatrix}
\frac{4}{6} & 0 \\
-\frac{1}{6} & 1
\end{bmatrix}^{-1} = \frac{6}{4} \begin{bmatrix}
1 & 0 \\
\frac{1}{6} & \frac{4}{6}
\end{bmatrix}
$$

Then

$$
MS = \frac{6}{4} \begin{bmatrix}
1 & 0 \\
\frac{1}{6} & \frac{4}{6}
\end{bmatrix} \begin{bmatrix}
\frac{3}{6} & \frac{1}{3} \\
\frac{1}{6} & \frac{2}{3}
\end{bmatrix} = \begin{bmatrix}
\frac{3}{4} & \frac{1}{4} \\
\frac{3}{24} & \frac{17}{24}
\end{bmatrix}.
$$

Thus, starting at 6, the probability to reach $\{4, 5\}$ is $\frac{1}{4}$ while starting at 7 the probability to reach this set is $\frac{17}{24}$.

(1g) From (f) and the computation of the invariant measure in part (c) we get that the probability the process is at 5 after a long time is $\frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$ if the process starts at 6 and it is $\frac{2}{3} \cdot \frac{17}{24} = \frac{17}{36}$ if it starts at 7.
(2a) This is a birth-death process with rates $\lambda_n = \frac{1}{n+1}$ for all $n \geq 0$, $\mu_n = 1$ for all $n \geq 1$, and $\mu_0 = 0$. Thus, for positive recurrence we compute
\[
\sum_{n=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = e < \infty.
\]
The convergence of this sum proves positive recurrence.

(2b) The invariant measure is given by the formula
\[
\pi(n) = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \pi(0) = \frac{\pi(0)}{n!}.
\]
Summing over $n$ we get $1 = \pi(0)e$. So $\pi(0) = e^{-1}$ and $\pi(n) = \frac{e^{-1}}{n!}$. The invariant measure is thus a Poisson distribution with parameter 1.

**Extra Credit:** Note that the rates at which the process leaves the various sites $n \geq 1$ is equal to one, for all of these sites. The rate at which the process leaves 0 is equal to $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. So the process leaves all sites at the same rate. Hence, we can replace the process by its discrete version, which is a process that jumps from 0 to a site $n \geq 1$ with probability $\frac{1}{2^n}$ and then from a site $n \geq 1$ it marches back to $n-1$, then $n-2$, and so on, until it reaches 0.

The invariant measure for the continuous-time and the discrete-time processes is the same. Hence, we can use what we know about the discrete-time Markov chains to solve the problem. Namely, the invariant measure solves
\[
\pi(n) = \frac{1}{2^n} \pi(0) + \pi(n+1) \quad \text{for all } n \geq 1
\]
(since sites that go to 0 are 0, with probability $1/2^n$, and $n+1$, with probability one) and $\pi(0) = \pi(1)$ (since only 1 goes to 0).

The above gives $\pi(n+1) - \pi(n) = -\frac{1}{2^n} \pi(0)$ and using telescoping sums we get
\[
\pi(n) - \pi(1) = -\pi(0) \sum_{k=1}^{n-1} \frac{1}{2^k} = -\pi(0) \left( \frac{1}{2} - \frac{1}{2^n} \right) = -\pi(0)(1 - \frac{1}{2^{n-1}}).
\]
Since $\pi(0) = \pi(1)$ we have that $\pi(n) = \frac{\pi(0)}{2^{n-1}}$ for all $n \geq 1$. Adding over all $n \geq 1$ we determine that $1 - \pi(0) = 2\pi(0)$ and so $\pi(0) = 1/3$. This gives $\pi(n) = \frac{1}{3} \cdot \frac{1}{2^{n-1}}$ for $n \geq 1$ and $\pi(0) = 1/3$.