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start by deriving the discrete version,
which is very simple:

Let S_n be the simple symmetric random walk starting at 0.
Let f:Z -> R be some given function defined on the integers.
Define, for an integer x and a time n>0:
g(n,x)=E[f(x+S_n)]

[basically, you start a simple sym random walk from x and you wait for time n, then look at f at the position of the walk. g is the average value of that.]

It should be easy to show that:
g(n+1,x) = (g(n,x+1)+g(n,x-1))/2 (*)
which implies that
g(n+1,x)-g(n,x) = (g(n,x+1)-2g(n,x)+g(n,x-1))/2
and of course g(0,x)=f(x) at all x.

This is the discrete version of the heat equation!
(\partial_t g = 0.5 \partial_{xx}^2 g with initial condition g(0,x)=f(x))

Can you see now how a random walker can solve the discrete heat equation?

It works pretty much the same way in the continuous setting, except that the "calculus" step you've done to get (*) must now be done using Ito's formula.

So start with B being BM starting from 0. Let f:R -> R be given.
We want to solve the heat equation with initial condition g(0,x)=f(x).
The suggested solution is g(t,x)=E[f(x+B(t))]
Now we just need to check that!!

That the initial condition is satisfied is immediate.

To check that g satisfies the heat equation, let x be fixed and define
Y(t)=f(x+B(t)).

Use Ito's formula to find a and b such that:
dY(t) = a(x+B(t)) dB(t) + b(x+B(t)) dt (**)

Let's see if you can do that yourself. If you get stuck, I can help. But it is essentially what we've done today!!! (note that x is fixed)

Once you got that part, then you take expected values of both sides of (**) and pretend that E[dY] = d E[Y]. This last step needs justification, but actually follows from standard probability theory and is not hard to show to be correct.
So let's pretend it is correct. So we have:

d E[f(x+B(t))] = E[a(x+B(t)) dB(t)] + E[b(x+B(t))] dt

But a(x+B(t)) depends B(t), while dB(t)=B(t+dt)-B(t) and thus is independent of B(t).

So E[a dB(t)] = E[a] E[dB(t)]. But E[dB(t)] = 0.

We now have
d g(t,x) = E[b(x+B(t))] dt

Since x was fixed, this means that
\partial_t g (t,x) = E[b(x+B(t))]

Now you'll need the form of b. If you did it right, you'll see that you're computing the average of a derivative. Just like before, pretend it is equal to the derivative of the average. You should get the heat equation now.

Next, we can derive the heat equation with boundary conditions.

let a<b be two integers. for x between a and b, let
g(x)=P(S_n+x reaches b before a)
this is the probability a random walk starting at x hits b before it hits a.
what are the properties of g(x)?

1) g(a)=0 and g(b)=1
2) if a<x<b, then g(x)=0.5 g(x+1) + 0.5 g(x-1)
(from x you go to either x-1 or x+1 and then your task is again to hit b before a!)
this can be rewritten as g(x+1)-2g(x)+g(x-1)=0, which is the Laplace equation in the discrete case (in the continuous case, it is: g''(x)=0.
this is the heat equation once the heat reaches equilibrium (i.e steady state, i.e. g(t,x) is now independent of time and is just g(x), and so derivative in time is gone)

so if we can find these probabilities, we can solve the following heat problem:
we keep the edge b at 1 degree and a at 0 degrees and we wait till the heat diffuses in the rod (a,b). g(x) is the heat at x.

a probabilistic interpretation (similar to the one in the first part you already worked out) is the following: to find the heat at x, let a random walker go from x and wander around till it reaches a or b. if it reaches b give it 1 point. if it reaches a give it 0 points. the average number of points it gets is the heat at x.

but now you can actually do that with any numbers at a and b! not just 0 and 1!
so if you keep b at B degrees and a at A degrees, then the heat at x
is g(x)= A P(S_n+x reaches a first) + B P(S_n+x reaches b first)
= E[f(S_T+x)] where T is the first time you exit (a,b) and
f is a function defined on the boundary of (a,b), i.e. at a and b, such that f(a)=A and f(b)=B.

(by the way, one can solve for g using induction, but if you think of it as being the probability of, starting at x, you hit b before a, then i guess you'd believe me if i tell you that the solution is: (b-x)/(b-a) !!!!)

now you can generalize this to more than one dimension!
(just like you can actually generalize the first part to more than one dimensions by using a multi-d Brownian motion to solve \partial g/partial t = 0.5 \Delta g with initial conditions g(0,x)=f(x) given.. the solution being g(t,x)=E[f(x+B(t))])

the general multi-d problem this time is:
you have a domain D and you are given the heat distribution at its boundary.
i.e. you are given a function f on the boundary of D.
you want to solve the steady-state heat equation: \Delta g=0 with boundary

condition: $g(x)=f(x)$ when x is on the boundary of D.

the solution is $g(x)=E[f(x+B(T))]$ where T is the first time the BM reaches the boundary. so you let a BM travel from x and you stop it when it gets to the boundary and look at the temperature at that point. the average temperature coming out of such a game is the temperature at x.