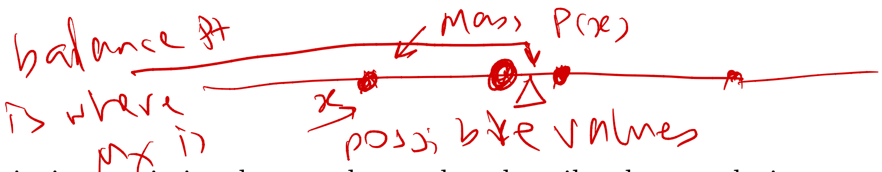
Lec11

3.3 Expected Values



As with a sample, there are descriptive statistics that can be used to describe the population.

Definition 11. Let X be a discrete random variable with set of possible values D and pmf $p(x)$. The **expected value** or **mean value** of X, denoted by $E[X]$ or μ_X, is

$$\sum_{x \text{ possible value}} x\,p(x)$$

Recall that, previously we use the **mean** as a measure of the center of the data set, i.e. the arithmetic average. But now, the **mean** refers to the center of the population.

Example 32. Consider a university having 15,000 students and let X = the number of courses for which a randomly selected student is registered. The pmf of X follows.

x	1	2	3	4	5	6	7
$p(x)$	0.01	0.03	0.13	0.25	0.39	0.17	0.02

Calculate the expected value of X, i.e $E(X)$.

Solution.

$$E[X] = 1\times 0.01 + 2\times 0.03 + 3\times 0.13 + \cdots = 4.57$$

Notice that μ here is not 4, the ordinary average of $1, \cdots, 7$, because the distribution puts more weight on 4, 5, and 6 than on other X values.

Example 33. Let $X = 1$ if a randomly selected vehicle passes an emissions test and $X = 0$ otherwise. Then X is a Bernoulli random variable with pmf $p(1) = p$ $p(0) = 1 - p$, from which $E(X) = p\times 1 + (1-p)\times 0 = p$. That is, the expected value of X is $E[X] = p$ = probability of 1.

The Expected Value of a Function

> If the random variable X has a set of possible values D and pmf $p(x)$, then the expected value of any function $h(X)$, denoted by $E[h(X)]$ is computed by
>
> $$E[h(x)] = \sum h(x) \cdot p(x)$$ over all possible value

Example 34. The cost of a certain vehicle diagnostic test depends on the number of cylinders X in the vehicle's engine. Suppose the cost function is given by $Y = h(X) = 20+3X+0.5X^2$. Calculate the expected value of Y. The pmf of X is as follows:

x	4	6	8
$p(x)$	0.5	0.3	0.2

y	40	56	76
$p(y)$	0.5	0.3	0.2

$h(x)$

Solution.

$$E[h(x)] = 40 \times 0.5 + 56 \times 0.3 + 76 \times 0.2$$
$$= 52$$

Example 35. A computer store has purchased three computers of a certain type at \$500 apiece. It will sell them for \$1000 apiece. The manufacturer has agreed to repurchase any computers still unsold after a specified period at \$200 apiece. Let X denote the number of computers sold, and suppose that $p(0) = 0.1$, $p(1) = 0.2$, $p(2) = 0.3$, and $p(3) = 0.4$. With $h(X)$ denoting the profit associated with selling X units, the given information implies that $h(X) = \text{revenue} - \text{cost} = 1000X + 200(3-X) - 1500 = 800X - 900$.

Find the expected profit.

Solution.

x	0	1	2	3
$p(x)$	0.1	0.2	0.3	0.4

$h(x)$	-900	-100	700	1500
$p(x)$	0.1	0.2	0.3	0.4

$$-900 \times 0.1 - 100 \times 0.2 + 700 \times 0.3 + 1500 \times 0.4$$
$$= 700$$

OR: $E[X] = 0 \times 0.1 + 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.4$
$= 2$

$h(X)$ is linear $\Rightarrow$ $h(2) = 1600 - 900$
$= 700$

Expected Value of a Linear Function

The $h(X)$ function of interest is quite frequently a linear function $aX + b$. In this case, $E[h(X)]$ is easily computed from $E(X)$ without the need for additional summation.

$$E(aX + b) = a E[X] + b$$

Proof.

$$E[aX+b] = \sum (ax+b)\, p(x)$$
$$= \sum (ax\, p(x) + b\, p(x))$$
$$= \sum ax\, p(x) + \sum b\, p(x)$$
$$= a E[X] + b \times 1$$

□

The Variance of X

We will use the variance of X to assess the amount of variability in (the distribution of) X, just as s^2 was used in Chapter 1 to measure variability in a sample.

Definition 12. Let X be a discrete random variable with pmf $p(x)$ and expected value μ. Then the **variance** of X, denoted by $Var(X)$, σ_X^2, is

$$E[(X - E[X])^2]$$

The **standard deviation** (SD) of X is

$$\sqrt{Var(X)}, \quad \sigma_X$$

Example 36. A library has an upper limit of 6 on the number of DVDs that can be checked out to an individual at one time. Consider only those who currently have DVDs checked out, and let X denote the number of DVDs checked out to a randomly selected individual. The pmf of X is as follows:

x	1	2	3	4	5	6
$p(x)$	0.3	0.25	0.15	0.05	0.1	0.15

Find the variance of X.

1) $E[X] = 1 \times 0.3 + 2 \times 0.25 + \cdots = 2.85$

2)

$(x-2.85)^2$	$(1-2.85)^2$	$(2-2.85)^2$	...
	0.3	0.25	...

$$Var(X) = (1-2.85)^2 \times 0.3 + (2-2.85)^2 \times 0.25$$

Solution.

$$+ \cdots = 3.2275$$

$$\sigma = \sqrt{3.2275} = 1.8$$

A Shortcut Formula for σ^2

$$\boxed{\mathrm{Var}(X) = E[X^2] - E[X]^2}$$

Proof.

$$Var(X) = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$$

$$= E[X^2] + E[-2\mu X] + E[\mu^2]$$

$$= E[X^2] - 2\mu E[X] + \mu^2$$

$$= E[X^2] - \mu^2$$

□

Example 37. (Example 36 continued)

1) $E[X] = 2.85$

x^2	1	4	9	16	25	36
$p(x)$	0.3	0.25	0.15	0.05	0.1	0.15

$Var(X) =$

$11.35 - 2.85^2$

$= 3.2275$

2) $E[X^2] = 1 \times 0.3 + 4 \times 0.25 + \cdots = 11.35$