Lec 8

2.5 Independence

$A \cap B = \emptyset$

$P(A \cap B) = P(\emptyset) = 0$

A & B indep.: $P(A)P(B) = 0 \Rightarrow P(A) = 0$ or $P(B) = 0$

We say A and B are independent events, meaning that info about one does not affect the prob. of the other.

Definition 6. Two events A and B are **independent** if

$$P(A \cap B) = P(A)\,P(B)$$

and are dependent otherwise.

NOTE: This definition implies if A and B are independent

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

A & B indep. $\Rightarrow P(A|B) = P(A)$

- $P(A|B) = P(A)$
- $P(B|A) = P(B)$

It is also straightforward to show that if A and B are independent, then so are the following pairs of events:

(1) $P(A|B^c) = P(A)$?

(2) $P(B|A^c) = P(B)$

(3) $P(A^c|B) = P(A^c)$

also $P(A^c|B^c) = P(A^c)$, $P(B^c|A) = P(B^c)$, $P(B^c|A^c) = P(B^c)$

$$P(A|B^c) = \frac{P(A \cap B^c)}{P(B^c)}$$

$P(A) - P(A \cap B)$

$1 - P(B)$

$$\frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{P(A) - P(A)P(B)}{1 - P(B)} = \frac{P(A)\cancel{(1 - P(B))}}{\cancel{1 - P(B)}} = P(A)$$

indep. of A & B

Example 19. Consider an experiment where the next car sold from a dealership is observed. Let C = a car has a CD player, M = a car has a manual transmission.

Given: $P(C) = 0.75, \quad P(M) = 0.15, \quad P(M \cup C) = 0.85.$

Question:

1. Are C and M mutually exclusive/disjoint events?

2. Are C and M independent events?

Solution.

$P(C \cap M) =$

$P(C \cup M) = P(C) + P(M) - P(C \cap M)$

$0.85 = 0.75 + 0.15 - P(C \cap M)$

$P(C \cap M) = 0.05 > 0$

$\Rightarrow C \cap M \neq \emptyset$

not disjoint

$0.05 \neq 0.75 \times 0.15$

not indep.

Example 20. (Exercise 74 from textbook page 89) The proportions of blood phenotypes in the U.S. population are as follows:

A	B	AB	O
0.40	0.11	0.04	0.45

Assuming that the phenotypes of two **randomly selected** individuals are **independent** of one another.

NOTE: randomly selected $\Longrightarrow$ independent

Question:

1. What is the probability that both phenotypes are O?

2. What is the probability that the phenotypes of two randomly selected individuals match?

Solution.

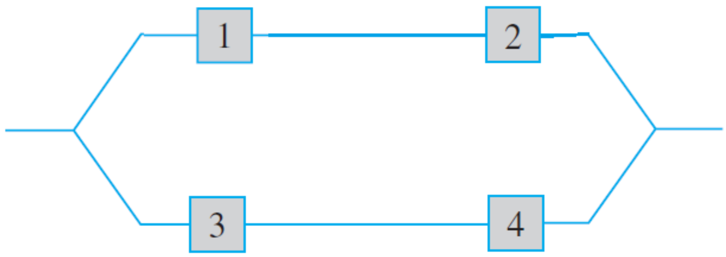
1) $P(O_1 \cap O_2) \overset{\text{indep.}}{=} P(O_1)P(O_2)$

$= 0.45 \times 0.45$

2) $P(\text{match}) = P(A_1 \cap A_2) + P(B_1 \cap B_2) + P(AB_1 \cap AB_2) + P(O_1 \cap O_2)$

$= 0.4^2 + 0.11^2 + 0.04^2 + 0.45^2$

Example 21. The entire system will work if either the top two components both work or the bottom two components both work. If components work independently of one another and $P(\text{component } i \text{ fails}) = 0.4$ for $i = 1, 2, 3, 4$.



Question:

1. What is the probability that all four components fails?

2. What is the probability that exactly one of the components fails?

3. What is the probability that at least one of the components fails?

4. What is the probability that at most one of the components fails?

5. What is the probability that the system works?

Solution.

Do it

3 Discrete Random Variables and Probability Distributions

3.1 Random Variables

Definition 7. A function, f, is a rule that takes an input value and returns an output.

For example, $y = x + 2$. Then if $x = 1$ then we have $y = 3$.

Definition 8. A random variable is

a function from sample space to the real numbers.

Example 22. Suppose our experiment is tossing a coin two times. Then the sample space is

$$S = \{HH, HT, TH, TT\}$$

If we define the random variable

$HH \to 2$, $HT \to 1$, $TH \to 1$, $TT \to 0$; $S \to \mathbb{R}$

$$X = \text{the number of heads you get in one experiment}$$

Then we can see that X can take values 0, 1, 2 .

NOTE: Random variables are usually denoted by uppercase letters, such as ______.

We will use lowercase letters to represent ______ of the corresponding random variable. The notation $X(\omega) = x$ means that x is a given possible value and we are interested in experiment out comes ω such that $X(\omega) = x$

We sometimes consider several different random variables from the same sample space.

Example 23. Two gas stations are located at a certain intersection. Each one has six gas pumps. Consider the experiment in which the number of pumps in use at a particular time of day is determined for each of the stations.

Define random variables X, Y, and U by

$X =$ the total number of pumps in use at the two stations
$Y =$ the difference between the number of pumps in use at station 1 and station 2
$U =$ the maximum of the numbers of pumps in use at the two stations

Station 1 had 2 pumps

If this experiment is performed and results $\omega = (2, 3)$, then

2 had 3 pumps

$$X = 2+3=5$$

$$Y = 2-3=-1$$

$$U = 3$$

Example 24. When a student calls a university help desk for technical support, he/she will either immediately be able to speak to someone (S, for success) or will be placed on hold (F, for failure). With $S = \{S, F\}$, define an random variable X by

$$X(S) = 1 \text{ and } X(F) = 0$$

The random variable X indicates ____

Example 25. Suppose a location in the United States is selected. Define the random variable Y by

$$Y = \text{ the height above sea level at the selected location}$$

Then the largest possible value of Y is 14,494 (Mt. Whitney), and the smallest possible value is 2282 (Death Valley). The set of all possible values of Y is the set of all numbers in the interval between 2282 and 14,494, that is, $[2282, 14494]$

and there are uncountably many (infinitely) values in this interval.

Lec9

Two Types of Random Variables

- ________________________________. If the possible outcomes of a random variable can be listed out using a finite (or countably infinite) set of single numbers (Example 22, 23, 24), then the random variable is discrete.
- ________________________________. If the possible outcomes of a random variable can only be described using an interval or union of intervals of real numbers (Example 25), then the random variable is continuous.

3.2 Probability Distributions for Discrete Random Variables

3.2.1 The Probability Mass Function

The **probability distribution** of X says how the total probability of 1 is distributed among the various possible X values. The probability distribution of X lists all possible values of X and their corresponding probabilities.