Math 1310

Review session

1. **(Optimization and Derivatives)**

   (a) You are to build a cone (with no base) of a fixed volume V with the minimum amount of material. What is the ratio of height to radius h/r of your cone? Remember the following formulas:

   - the volume of a cone is $\frac{1}{3}\pi r^2 h$;
   - the surface area of a cone without base is $\pi r\sqrt{r^2+h^2}$.

Solution: Since V is fixed, we can see $V = \frac{1}{3}\pi r^2 h$ as a relation between h and r. Hence we can express h in terms of r as follows

$$r^2 = \frac{3V}{\pi h}.$$

Thus we get that the surface is

$$S(h) = \pi\sqrt{\frac{3V}{\pi h}}\sqrt{\frac{3V}{\pi h}+h^2}.$$

To find the value of h giving the minimum surface, we can minimize $S^2(h)$ as well. Thus we get to study

$$S^2(h) = \pi^2\frac{3V}{\pi h}\left(\frac{3V}{\pi h}+h^2\right) = 3V\pi\left(\frac{3V}{\pi h^2}+h\right).$$

Thuse we have

$$(S^2(h))' = 3V\pi\left(-\frac{6V}{\pi h^3}+1\right).$$

If we set it equal to zero, we get the equation

$$\frac{6V}{\pi h^3} = 1,$$

which leads to

$$h = \sqrt[3]{\frac{6V}{\pi}}.$$

Now, $r = \sqrt{\frac{3V}{\pi h}}$, thus we get

$$\frac{h}{r} = \sqrt[3]{\frac{6V}{\pi}}\sqrt{\frac{\pi h}{3V}} = \sqrt[3]{\frac{6V}{\pi}}\sqrt{\frac{\pi\sqrt[3]{\frac{6V}{\pi}}}{3V}}.$$

2. **(Integrals)**

Comupute the following integrals.

(a) $\int_1^\infty 3x^2e^{-3x^3}dx$

Solution: We notice that the polynomial $3x^2$ is the derivative of x^3. Hence if we set $u = x^3$, we get $du = 3x^2dx$. By definition of improper integral we have

$$\int_1^\infty 3x^2e^{-3x^3}dx = \lim_{t\to\infty}\int_1^t 3x^2e^{-3x^3}dx \tag{1}$$

So we first compute the proper integral $\int_1^t 3x^2e^{-3x^3}dx$ and use substitution. We see that for $x = 1$, then $u = 1$, while for $x = t$ we get $u = t^3$. Hence we get

$$\int_1^t 3x^2e^{-3x^3}dx = \int_1^{t^3} e^{-3u}du \tag{2}$$

Now, we know that $-\frac{e^{-3u}}{3}$ is an antiderivative of e^{-3u}, so we get

$$\int_1^{t^3} -e^{-3u}du = [-\frac{e^{-3u}}{3}]_1^{t^3} = \frac{1}{3e^3} - \frac{e^{-3t^3}}{3} \tag{3}$$

Hence we get

$$\int_1^\infty 3x^2e^{-3x^3}dx = \lim_{t\to\infty}\left(\frac{1}{3e^3} - \frac{e^{-3t^3}}{3}\right) = \frac{1}{3e^3} \tag{4}$$

(b) $\int_0^\infty xe^{-3x}dx$

Solution: We first use integration by parts in $\int_0^t xe^{-3x}dx$ and then take the limit as t goes to ∞ to calculate the improper integral. We see that it is easy to take an antiderivative of e^{-3x} and if we differentiate x we are left with 1. Hence in integration by parts we integrate e^{-3x} (the antiderivative is $-\frac{e^{-3x}}{3}$) and differentiate x. I.e, in the u and v notation (see page 383 of the book) $u = x$ and $dv = e^{-3x}$:

$$\int_0^t xe^{-3x}dx = \left[-\frac{xe^{-3x}}{3}\right]_0^t - \int_0^t -\frac{e^{-3x}}{3}dx \tag{5}$$

Now we study separately the second summand, i.e.

$$-\int_0^t -\frac{e^{-3x}}{3}dx = \frac{1}{3}\int_0^t e^{-3x}dx = \frac{1}{3}\left[-\frac{e^{-3x}}{3}\right]_0^t \tag{6}$$

Hence we get

$$\int_0^t xe^{-3x}dx = -\frac{te^{-3t}}{3} - \frac{e^{-3t}}{9} + \frac{1}{9} \tag{7}$$

Hence to compute the improper integral we are left with two limits.

We have

$$\lim_{t\to\infty} \frac{e^{-3t}}{9} = 0 \tag{8}$$

and

$$\lim_{t\to\infty} te^{-3t} = \frac{t}{e^{3t}} \overset{\text{H}}{=} \lim_{t\to\infty} \frac{1}{3e^{3t}} = 0 \tag{9}$$

(c) Find an antiderivative for $\frac{1}{2\sqrt{x+3}+x}$

Solution: The problem is well stated just for $x > -3$, since we are taking the square root of $x+3$. So from now on we will assume $x > -3$. Now, we want to calculate the integral

$$\int \frac{1}{2\sqrt{x+3}+x}dx \tag{10}$$

First we do substitution to get rid of the square root. If we set $y = \sqrt{x+3}$, then $y^2 = x+3$ and hence $2ydy = dx$ If we do this substitution in the integral, we get

$$\int \frac{1}{2\sqrt{x+3}+x}dx = \int \frac{2y}{2y+y^2-3}dy \tag{11}$$

We notice that we can factor $y^2+2y-3 = (y-1)(y+3)$. Now we want to write

$$\frac{2y}{2y+y^2-3} = \frac{A}{y-1} + \frac{B}{y+3} = \frac{A(y+3)+B(y-1)}{(y+3)(y-1)} \tag{12}$$

This leads to the system

$$\begin{cases} Ay + By = 2 \\ 3A - B = 0 \end{cases} \tag{13}$$

From the second equation we get $B = 3A$; sustituting in the first one we have $3A + A = 2$. Hence we have $A = \frac{1}{2}$ and $B = \frac{3}{2}$. This tells us that it's enough to compute the integrals

$$\int \frac{1}{2(y-1)}dy, \quad \int \frac{3}{2(y+3)}dy \tag{14}$$

We know that an antiderivative of $\frac{1}{x}$ is $\ln|x|$, and using this idea we have

$$\int \frac{1}{2(y-1)}dy = \frac{1}{2}\int \frac{1}{y-1}dy = \frac{1}{2}\ln|y-1| \tag{15}$$

and

$$\int \frac{3}{2(y+3)}dy = \frac{3}{2}\int \frac{1}{y+3}dy = \frac{3}{2}\ln|y+3| \tag{16}$$

Now in such functions we have to substitute back $y = \sqrt{x+3}$ and hence we get that a possible antiderivative of the function we started with is

$$\frac{1}{2}\ln|\sqrt{x+3} - 1| + \frac{3}{2}\ln|\sqrt{x+3} + 3| \tag{17}$$

3. **(Limits)** Compute the following limits

(a) $\lim_{x\to\infty} \sqrt{3x^2+8x+6} - \sqrt{3x^2+3x+1}$

Solution: We first manipulate the expression

$$\begin{aligned}
&\sqrt{3x^2+8x+6} - \sqrt{3x^2+3x+1} \\
&= (\sqrt{3x^2+8x+6} - \sqrt{3x^2+3x+1})\frac{\sqrt{3x^2+8x+6} + \sqrt{3x^2+3x+1}}{\sqrt{3x^2+8x+6} + \sqrt{3x^2+3x+1}} \\
&= \frac{3x^2+8x+6-3x^2-3x-1}{\sqrt{3x^2+8x+6} + \sqrt{3x^2+3x+1}} \\
&= \frac{5x+5}{\sqrt{3x^2+8x+6} + \sqrt{3x^2+3x+1}} \\
&= \frac{5x+5}{\sqrt{3}x\left(1+\frac{8}{\sqrt{3}x}+\frac{6}{\sqrt{3}x^2}\right) + \sqrt{3}x\left(1+\frac{3}{\sqrt{3}x}+\frac{1}{\sqrt{3}x^2}\right)} \\
&= \frac{5+\frac{5}{x}}{\sqrt{3}\left(1+\frac{8}{\sqrt{3}x}+\frac{6}{\sqrt{3}x^2}\right) + \sqrt{3}\left(1+\frac{3}{\sqrt{3}x}+\frac{1}{\sqrt{3}x^2}\right)}
\end{aligned} \tag{18}$$

Now we see

$$\begin{aligned}
&\lim_{x\to\infty} \frac{5}{x} = 0 \\
&\lim_{x\to\infty} 1 + \frac{3}{\sqrt{3}x} + \frac{1}{\sqrt{3}x^2} = 1 \\
&\lim_{x\to\infty} 1 + \frac{8}{\sqrt{3}x} + \frac{6}{\sqrt{3}x^2} = 1
\end{aligned} \tag{19}$$

and thus we get

$$\lim_{x\to\infty} \sqrt{3x^2+8x+6} - \sqrt{3x^2+3x+1} = \frac{5}{2\sqrt{3}} \tag{20}$$

(b) $\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$

Solution: We have

$$\begin{aligned}
\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) &= \\
&= \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}}
\end{aligned} \tag{21}$$

$$
\begin{aligned}
&\overset{\text{H}}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right)\left(-\frac{\pi}{x^2}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \pi cos\left(\frac{\pi}{x}\right) = \pi
\end{aligned} \tag{22}
$$

(c) $\lim_{x\to 0^+} \sin x \ln x$

Solution: We know that if $\lim_{x\to a} f(x) = c$ and $\lim_{x\to a} g(x) = d$, then we have $\lim_{x\to a} f(x)g(x) = cd$ (see page 104 for a review of such properties). Then we can write $\sin x \ln x = \left(\frac{\sin x}{x}\right)(x \ln x)$. We know that $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$ (if you do not remember it, check it with l'Hospital's Rule). Then it is enough to study $\lim_{x\to 0^+} x \ln x$. To this aim, we use l'Hospital's Rule

$$
\begin{aligned}
\lim_{x\to 0^+} x \ln x &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \\
&\overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to 0^+} -x = 0
\end{aligned} \tag{23}
$$

Hence $\lim_{x\to 0^+} \sin x \ln x = \left(\lim_{x\to 0^+} \frac{\sin x}{x}\right)\left(\lim_{x\to 0^+} x \ln x\right) = 1 \cdot 0 = 0.$