Math 1310 Lab 6. (Sec 3.3-3.5)

Name/Unid: ________________________________ Lab section: ____

1. **(Play with differential rules, get trigonometric identities.)**

   Question 1. Differentiate the identity $\cos 2x = \cos^2 x - \sin^2 x$ with respect to x. Derive a formula for $\sin 2x$. **(2 pts)**

   Question 2. Differentiate the identity $\sin 2x = 2 \sin x \cos x$ with respect to x. Derive a formula for $\cos 2x$. **(2 pts)**

Question 3. We know that $\sin 3x = 3\sin x - 4\sin^3 x$. Differentiate it with respect to x, and apply the identity $\cos^2 x + \sin^2 x = 1$, to get a formula for $\cos 3x$. **(2 pts)**

Solution: A1. $-2\sin 2x = -2\cos x \sin x - 2\sin x \cos x = -4\sin x \cos x$. So $\sin 2x = 2\sin x \cos x$. A2. $2\cos 2x = 2\cos^2 x - 2\sin^2 x$. So $\cos 2x = \cos^2 x - \sin^2 x$. A3. $3\cos 3x = 3\cos x - 12\sin^2 x \cos x = -9\cos x + 12\cos^3 x$. Therefore, $\cos 3x = 4\cos^3 x - 3\cos x$.

2. **(All roads lead to Rome.)**

We'd like to differentiate $y = f(x) = \frac{1}{(x^2+1)(x^2+2)}$ in four different ways.

Question 1. Composite function approach, I. Let $a(x) = \frac{1}{x}$, $b(x) = (x+1)(x+2) = x^2 + 3x + 2$, $c(x) = x^2$, so you can see $f(x) = a \circ b \circ c(x)$. If we differentiate it with respect to x, we have $f'(x) = a'(b(c(x))) \cdot b'(c(x)) \cdot c'(x)$ (This is the chain rule for three functions). Explicitly compute $f'(x)$ using this strategy. **(3 pts)**

Question 2. Composite function approach, II. Since $(x^2+1)(x^2+2) = x^4 + 3x^2 + 2$, We consider $A(x) = \frac{1}{x}$, $B(x) = x^4 + 3x^2 + 2$, and $f(x) = A \circ B(x)$. Apply the chain rule for two functions to get $f'(x)$. Explicitly compute $f'(x)$ using this strategy. **(3 pts)**

Question 3. Product rule. Let $C(x) = \frac{1}{x^2+1}$, $D(x) = \frac{1}{x^2+2}$. Then we have $f(x) = C)(x)D(x)$. Use the product rule with respect to C and D to get $f'(x)$. Explicitly compute $f'(x)$ using this strategy. **(3 pts)**

Question 4. Implicit differentiation. Set $y = \frac{1}{(x^2+1)(x^2+2)}$, so we have $y \cdot (x^2+1) \cdot (x^2+2) = 1$. Now differentiate with respect to x, to get an equality involving y' and solve it for y' (which is our $f'(x)$). Remember that $y = \frac{1}{(x^2+1)(x^2+2)}$. Explicitly compute $f'(x)$ using this strategy. **(3 pts)**

Solution: A1. $f'(x) = a'(b(c(x))) \cdot b'(c(x)) \cdot c'(x) = \frac{-1}{((x^1+1)(x^2+2))^2} \cdot (2x^2+3) \cdot (2x)$.
A2. $f'(x) = A'(B(x)) \cdot B'(x) = \frac{-1}{(x^4+3x^2+2)^2} \cdot (4x^3+6x)$.
A3. $f'(x) = C(x)D'(x) + C'(x)D(x) = \frac{1}{x^2+1} \cdot \frac{-1}{(x^2+2)^2} \cdot (2x) + \frac{1}{x^2+2} \cdot \frac{-1}{(x^2+1)^2} \cdot (2x)$
$= \frac{-1}{((x^1+1)(x^2+2))^2} \cdot (2x^2+3) \cdot (2x)$.
A4. Solve for y' to get $\frac{-1}{((x^1+1)(x^2+2))^2} \cdot (2x^2+3) \cdot (2x)$.

3. **(Application of derivatives)**

An artist has created an ice sculpture representing the world. It is an ice sphere of radius $1m$. Because of the temperature of the room, the sculpture is slowly melting. Assume that the function describing the radius is given by $r(t) = A(1 - e^{\alpha t - \beta})$, where t is measured in hours and A, α and β are constants. Also, the sculpture will take exactly 10 hours to melt and the rate at which the radius is decreasing at $t = 0$ is $-\frac{e^{-10}}{1-e^{-10}} m/s$.

Question 1 Determine the function $r(t)$. Remember that it has to describe the radius of the sphere for the time $0 \leq t \leq 10$ (the domain given by the physics of the problem). Also, what function gives you the rate of change given by the data? **(3 pts)**

Question 2 Determine the surface area of the sculpture and provide the function describing the rate at which the area of the surface is decreasing. **(2 pts)**

Question 3 Determine the volume of the sculpture and provide the function describing the rate at which the volume is decreasing. **(2 pts)**

Solution:

(a) $\beta = 10$ was given in class. Since at time 10 the radius has to be 0, we have $A(1 - e^{\alpha \cdot 10 - 10}) = 0$. Since the radius for $t < 10$ is not 0, A is not zero. So we have the condition $1 - e^{\alpha \cdot 10 - 10} = 0$. Since $e^x = 1$ if and only if $x = 0$, we get $10\alpha = 10$. Hence $\alpha = 1$. Now to determine A we just use the fact that for $t = 0$ the radius has to be 1. Plugging in 0 in $r(t)$ we get $1 = A(1 - e^{-10})$. So $A = \frac{1}{1-e^{-10}}$. This gives us that the function describing the radius is

$$r(t) = \frac{1 - e^{t-10}}{1 - e^{-10}} \tag{1}$$

(b) The surface area of a sphere is given by $4\pi r^2$. Hence we have to take the derivative with respect to time of $S(t) = 4\pi \left(\frac{1-e^{t-10}}{1-e^{-10}}\right)^2$. We get

$$S'(t) = \frac{8\pi}{(1 - e^{-10})^2}(1 - e^{t-10})(-e^{t-10}) \tag{2}$$

(c) The volume of a sphere is given by $\frac{4}{3}\pi r^3$. Hence we have to take the derivative with respect to time of $V(t) = \frac{4}{3}\pi \left(\frac{1-e^{t-10}}{1-e^{-10}}\right)^3$. We get

$$V'(t) = \frac{4\pi}{(1 - e^{-10})^3}(1 - e^{t-10})^2(-e^{t-10}) \tag{3}$$