

Math 1310 Lab 5.

Name/Unid: \_\_\_\_\_ Lab section: \_\_\_\_\_

1. **(Evaluating derivatives using the definition)(See Section 2.7)** For each of the following functions, do the following steps. First, determine the domain and sketch the graph of the function given  $f$ . Second, evaluate its derivative **using the definition** and determine the domain of the derivative function  $f'$  (the graph of  $f$  might help you for this aim). Last, sketch the graph of the derivative function  $f'$  on a separate Cartesian plane.

(a)  $\sqrt{x^2 - 6x + 9}$  (2 pts)

(b)  $\sqrt{x^2 + 3}$  (2 pts)

Recall that a function  $f$  is called even if  $f(-x) = f(x)$  for all  $x$  in its domain and odd if  $f(-x) = -f(x)$  for all such  $x$  (e.g.  $\cos(x)$  is an even function, while  $\sin(x)$  is an odd one). Prove each of the following. Now assume that you are given a function  $f$  which is defined for all real numbers and which is everywhere differentiable. Then, **using the definition**, prove the following:

- (c) if  $f$  is an even function, then  $f'$  is an odd function; **(2 pts)**  
 (d) if  $f$  an odd function, then  $f'$  is an even function. **(2 pts)**

Hint: since you have information about  $f(-x)$ , what does it tell you about  $f(-x+h)$ ? Also, if you know  $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ , what can you deduce about  $\lim_{h \rightarrow 0} \frac{f(x-h)-f(x)}{h}$ ?

**Solution:**

- (a)  $\sqrt{x^2 - 6x + 9} = \sqrt{(x-3)^2} = |x-3|$ , so the domain is all the real numbers. The graph is given by the graph of  $|x|$  shifted by 3 units to the right. This already tells us that at 3 the function is not differentiable. Assuming  $x > 3$ , then  $|x-3| = x-3$  and then

$$\lim_{h \rightarrow 0} \frac{(x+h-3) - (x-3)}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.$$

Assuming  $x < 3$ , then  $|x-3| = -x+3$  and then

$$\lim_{h \rightarrow 0} \frac{(-x-h+3) - (-x+3)}{h} = \lim_{h \rightarrow 0} \frac{-h}{h} = -1.$$

- (b)  $x^2+3 > 0$  for any  $x$ , since 3 is positive and  $x^2$  is nonnegative. Hence the domain is all of the real numbers. Now we have

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)^2+3} - \sqrt{x^2+3}}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2+3 - (x^2+3)}{h(\sqrt{(x+h)^2+3} + \sqrt{x^2+3})} \\ &= \lim_{h \rightarrow 0} \frac{2xh+h^2}{h(\sqrt{(x+h)^2+3} + \sqrt{x^2+3})} \\ &= \lim_{h \rightarrow 0} \frac{2x+h}{\sqrt{(x+h)^2+3} + \sqrt{x^2+3}} \\ &= \frac{x}{\sqrt{x^2+3}}. \end{aligned}$$

This shows that the domain of the derivative is all of the real numbers too.

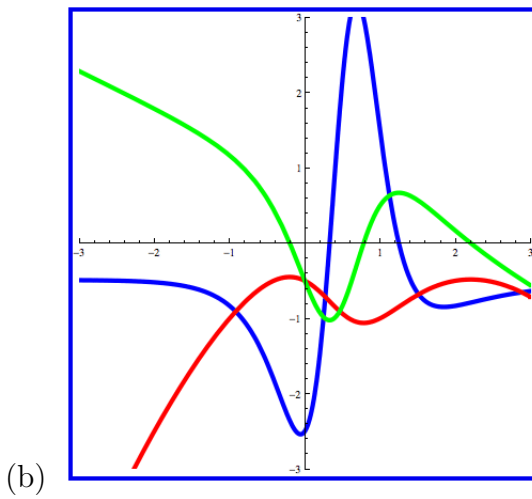
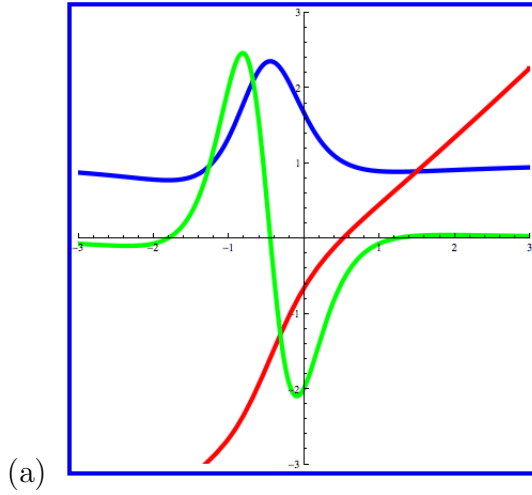
(c) We assume  $f$  is even, and hence  $f(-x) = f(x)$  for any  $x$ . Then we want to show  $f'(-x) = -f'(x)$ . So we evaluate the derivative at  $-x$  and try to express it in terms of  $f'(x)$ . We start evaluating  $f'$  at the point  $-x$  using the definition

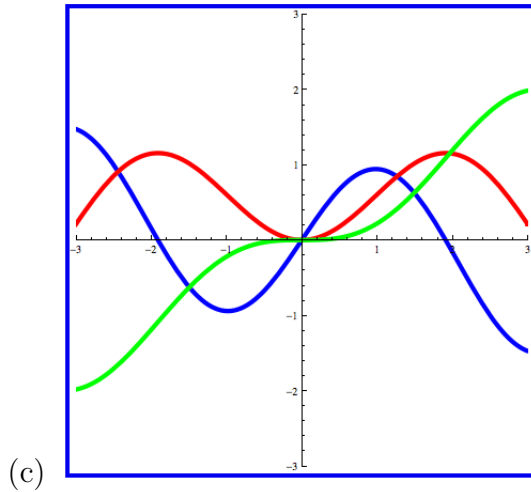
$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} &= \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h} && f \text{ even} \\ &= \lim_{h \rightarrow 0} -\frac{f(x-h) - f(x)}{-h} \\ &= -\lim_{k \rightarrow 0} \frac{f(x+k) - f(x)}{k} && k = -h \\ &= -f'(x) \end{aligned}$$

(d) We assume  $f$  is odd, and hence  $f(-x) = -f(x)$  for any  $x$ . Then we want to show  $f'(-x) = f'(x)$ . So we evaluate the derivative at  $-x$  and try to express it in terms of  $f'(x)$ . We start evaluating  $f'$  at the point  $-x$  using the definition

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} &= \lim_{h \rightarrow 0} \frac{-f(x-h) + f(x)}{h} && f \text{ odd} \\ &= \lim_{h \rightarrow 0} -\frac{f(x-h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h} \\ &= \lim_{k \rightarrow 0} \frac{f(x+k) - f(x)}{k} && k = -h \\ &= f'(x) \end{aligned}$$

2. (Understanding derivatives from a graph)(See section 2.7) In the following graphs, determine which curves are the original function, and which are its first and second derivative. For each graph, provide an explanation in words of your choice. (6 pts)





**Solution:**

- (a) Red is the original function. Blue is the first derivative. Green is the second derivative.
- (b) Red is the original function. Green is the first derivative. Blue is the second derivative.
- (c) Green is the original function. Red is the first derivative. Blue is the second derivative.

3. (Applications of derivatives)

(a) You are driving your car at a constant speed  $v = 20m/s$ . This leads to a gasoline consumption of 1 liter (1l) each 15 kilometers. Assume that you start your journey with 80 liters of gasoline in the tank of your car and that you drive until you are out of fuel and answer the following questions (**remember to include the units of measure!!!**):

- 1 how many kilometers will have you driven when you run out of fuel? **(1 pt)**
- 2 What is the rate of consumption of the gasoline during your trip? Hint: since you are travelling at a constant speed, you can say how many liters per second you need. **(1 pt)**

- (b) If a stone is thrown vertically upwards from the surface of the earth with a velocity of  $15m/s$ , its height  $h$  (in meters) after  $t$  seconds is given by the function

$$h(t) = (15m/s)t - (9.81m/s^2)t^2.$$

Keeping in mind that the velocity is the derivative of the position, answer the following questions (**remember to include the units of measure!!!**):

- 1 what is the speed of the stone after  $0.5s$ ? (**1 pt**)
- 2 What is the maximum height the stone reaches? (**1 pt**)
- 3 How long does the stone take to fall back onto the ground? (**1 pt**)

**Solution:**

- (a) You are moving at a speed of  $20m/s$  and you need one liter for covering  $15km = 15000m$ . So you take  $(15000m)/(20m/s) = 750s$  seconds to cover  $15km$  and hence to use 1 liter. So the rate of consumption of gasoline is  $1/750l/s$ . Since you have 80 liters and you need one to cover  $15km$ , you will have covered  $80l \cdot 15km/l = 1200km$ .
- (b) The derivative is  $v(t) = 15m/s - (19.62m/s^2)t$ , so for  $t = 0.5s$  we get  $v(0.5) = 5.19m/s$ . The maximum height is reached when the velocity is zero, so we solve the equation  $15m/s - (19.62m/s^2)t = 0m/s$  and we get  $t = 0.765s$ . If we plug it into the formula for position, we get  $h(0.765) = 5.733m$ . The time the stone takes to fall back onto the ground is twice as much as the time it needs to reach the maximum height. Hence it is  $1.53s$ . You can check it is the right answer also by solving the quadratic equation  $(15m/s)t - (9.81m/s^2)t^2 = 0m$ .

4. **(Some computations)**(See Sections 3.1-3.2) Using the properties in sections 3.1-3.2, compute the following derivatives.

(a)  $\frac{\sqrt{x}}{e^x+1}$  **(2 pts)**

(b)  $\frac{e^x-1}{3x^2}$  **(2 pts)**

(c)  $e^{-x}$  **(2 pts)**



**Solution:**

- (a) We know that the derivative of a sum is the sum of the derivatives, so the derivative of  $e^x + 1$  is  $e^x$ . Also, we know that the derivative of  $\sqrt{x}$  is  $\frac{1}{2\sqrt{x}}$ , so putting things together we get

$$\frac{\frac{1}{2\sqrt{x}}(e^x + 1) - \sqrt{x}e^x}{(e^x + 1)^2}$$

- (b) By the rule for polynomials, we know that the derivative of  $x^2$  is  $2x$  and hence, since we are multiplying by a constant, the derivative of  $3x^2$  is  $6x$ . Also, we have that the derivative of  $e^x - 1$  is  $e^x$ , since we take the difference of the two derivatives. Putting these things together in the quotient formula, we get

$$\frac{3e^x x^2 - (e^x - 1)6x}{9x^4}$$

- (c) We have  $e^{-x} = \frac{1}{e^x}$ . The derivative of  $e^x$  is  $e^x$ , so using the quotient rule, we have

$$-\frac{e^x}{(e^x)^2} = -\frac{1}{e^x} = -e^{-x}$$