Math 1310 Lab 4. (Sec 2.4-2.7)

Name/Unid: _____ Lab section: _____

1. **(The continuity of a composite function)** Recall Theorem 9, Sec. 2.4: If $g(x)$ is continuous at a, and $f(x)$ is continuous at $g(a)$, then $(f \circ g)(x)$ is continuous at a. (You may also find Theorem 4, 7 of Sec 2.4 useful for this problem)

   Question 1. Use Theorem 9, Sec 2.4 to show that $F(x) = [x^2]$ is continuous at $x = 1.5$. **(2 pts)**

   Question 2. Use Theorem 9, Sec 2.4 to show that $G(x) = \tan\left(\frac{1}{x^2+1}\right)$ is continuous for all $x \in \mathbb{R}$, even though $\tan x$ is not continuous everywhere on $\mathbb{R}$. (Hint. Show that $0 < \frac{1}{x^2+1} \le 1 < \frac{\pi}{2}$ for all $x \in \mathbb{R}$.) **(2 pts)**

Question 3. Use Theorem 9, Sec 2.4 to show that $H(x) = \ln{(\sin x + 2)}$ is continuous for all $x \in \mathbb{R}$, even though $\ln x$ is not defined for $x \leq 0$. **(2 pts)**

Solution: Write F, G, H in the form of $f \circ g$. First check that g is continuous on $\mathbb{R}$, and then check the range of g is included in the domain of f.

2. (**Polynomial interpolation**) A function f is defined as follows: $f(x) = \begin{cases} e^x & \text{if } x \le 0 \\ e^{1-x} & \text{if } x \ge 1. \end{cases}$

As you can see, this function is not defined for $0 < x < 1$. We'd like to do some "surgery" on this undefined interval to make this function smooth.

Recall Definition 3, Sec. 2.7: a function f is differentiable at a if f $f'(a)$ exists.

Question 1. Make a plot of f on $(-\infty, 0] \cup [1, \infty)$. (**2 pts**)

Question 2. (Linear interpolation) We'd like to define $f(x) = ax + b$ on $x \in (0, 1)$ for some constants a, b, to make f continuous on $\mathbb{R}$. What are a and b? (**2 pts**)

Question 3. (Linear interpolation is not good enough) Argue that f is not differentiable at $x = 0$ for the f constructed in Question 2. (Assume that you know the left-derivative of f at $x = 0$ is 1. Why not take a look at figure 13, Sec 1.5?) **(2 pts)**

Question 4. (Quadratic interpolation) Define $f(x) = -x^2 + x + 1$ on $x \in (0, 1)$. **Show that the left-derivative f at $x = 1$ equals -1, by showing the following identity holds:** (See also example 3, Sec 2.7) **(2 pts)**

$$\lim_{h \to 0} \frac{(-(x+h)^2 + (x+h) + 1) - (-x^2 + x + 1)}{h} = -2x + 1.$$

Along with the fact that the right derivative of f at 1 is -1 (you do not need to show this), f is differentiable at 1. Actually, for those who are interested, it can be shown that f is everywhere differentiable on $\mathbb{R}$.

Remarks. Although the f constructed in Question 4. is differentiable everywhere, it is still not twice-differentiable at $x = 0, 1$. So, to guarantee the higher-order derivatives of f exist, we have to choose higher-order polynomials interpolated in the interval.

Solution: A2. $a = 0, b = 1$. A3. The left-derivative of f at 0 is 1, while the right-derivative of f at 0 is 0.

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3. **(Different levels of smoothness of a function.)**

We'd like to construct different functions $f : (-1, 1) \to \mathbb{R}$ with different properties below. (Recall Theorem 4. in Sec 2.7 that if a function is differentiable at some point, then it is continuous at that point. If a function can differentiate many times, it is considered quite smooth.)

(a) $f(x)$ is differentiable everywhere on $(-1, 1)$.
(b) $f(x)$ is differentiable everywhere on $(-1, 1)$ except at $x = 0$. But $f(x)$ is still continuous everywhere on $(-1, 1)$.
(c) $f(x)$ is continuous everywhere on $(-1, 1)$, however, it is differentiable only at $x = 0$.
(d) $f(x)$ is continuous everywhere on $(-1, 1)$, but it is nowhere differentiable on $(-1, 1)$.
(e) $f(x)$ is differentiable at $x = 0$. But other than $x = 0$, $f(x)$ is not even continuous at every single point in $(-1, 1)$.
(f) $f(x)$ is continuous at $x = 0$, but not differentiable at $x = 0$. $f(x)$ is not continuous at every single point in $(-1, 1)$ except 0.
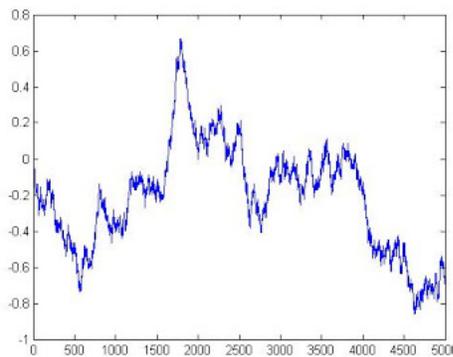(g) $f(x)$ is nowhere continuous on $(-1, 1)$.

Question 1. Construct 2 examples of f for case (a). **(2 pts)**

Question 2. Construct 1 example of f for case (b). **(2 pts)**

Case (c) and (d) are obviously harder. There are two famous examples for case (d): The first example is the **Weierstrass function**,
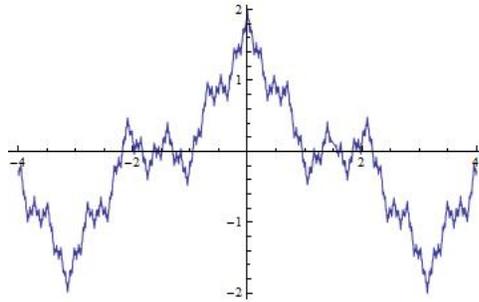
And the second example for case (d) is any 1-dimensional Brownian motion path.

As you can see, these functions are quite rough, aren't they?

To construct an example for (c), let $W(x)$ be the Weierstrass function, and the function $f(x) = x^2 W(x)$ is what we want.

Question 3. Show that $f(x) = x^2 W(x)$ is differentiable at $x = 0$. Hint: here you can use this fact for Weierstrass functions $W(x)$: $-2 \leq W(x) \leq 2$ for all $x \in (-1, 1)$. That means we have to show the limit

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h \to 0} \frac{h^2 W(h)}{h} = \lim_{h \to 0} h W(h)$$

equals 0. **Why is this limit** 0**? (3 pts)**

It's harder to show $f(x) = x^2 W(x)$ is not differentiable elsewhere, of which the proof I would like to neglect here. But it's not hard to show $f(x) = x^2 W(x)$ is continuous on $(-1, 1)$!

Question 4. Assume that you know $W(x)$ is continuous on $x \in (-1, 1)$. Then why $f(x) = x^2 W(x)$ is continuous on $(-1, 1)$? **(2 pts)**

For case (e), consider $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ -x^2 & \text{if } x \text{ is irrational} \end{cases}$. Those who feel interested may work out this case by themselves. Case (f) and (g) are similar, and you only need to do part of case (g). For case (f), we define $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.

Question 5. For case (g), we consider $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$. Show that f is not continuous at $x = 0$. (Hint: let $x_1 = 1$, $x_2 = 1/2$, $x_3 = 1/3$, and so forth. What are the values for $f(x_n)$? If, instead, we let $y_1 = \pi$, $y_2 = \pi/2$, $y_3 = \pi/3$, then what's going on for $f(y_n)$?)(Similarly, you can show f is not continuous elsewhere.) **(2 pts)**

Solution: A1. Examples are: $f(x) = 23, x^2 - 7, \sin x, e^x$. A2. $f(x) = |x|$. A3. Squeeze theorem. A4. The product of two continuous functions are continuous. A5. $\{f(x_n)\} \to 1$, while $\{f(y_n)\} \to 0$.