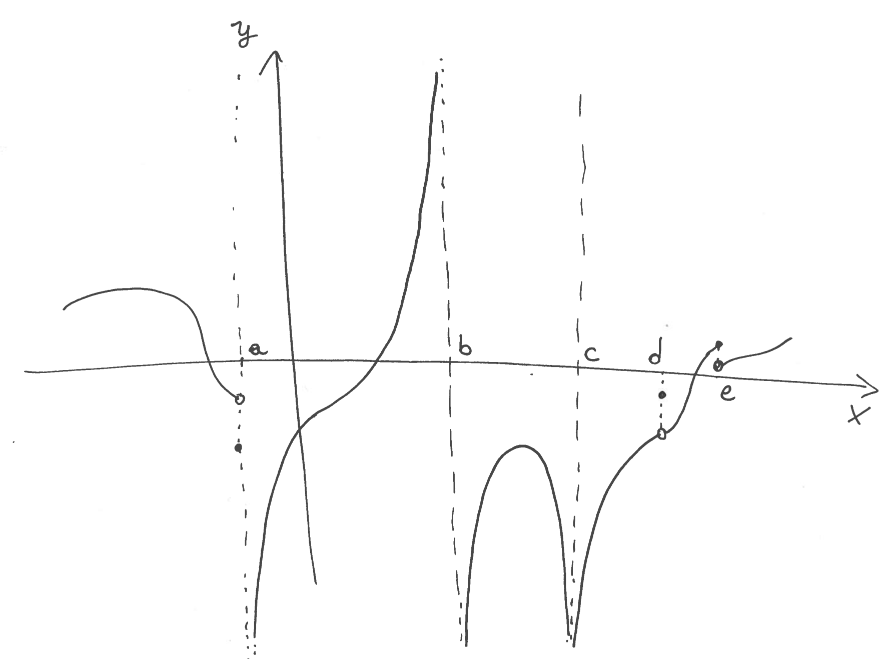
Math 1310 Lab 3.

Name/Unid: ________________________________ Lab section: ____

1. **(Understanding limits from a graph)** Given a function f, the concept of the limit $\lim_{x\to x_0} f(x)$ describes what happens to the values $f(x)$ takes as x gets closer and closer to x_0. Anyway, we neglect what is the actual value $f(x_0)$ (it could even be not defined!). If the values of $f(x)$ get closer and closer to a unique number λ, then we say that the limit as x approaches x_0 exists and has value λ. Indeed the limit, when is defined, is a number, which has the property of "capturing" what happens around our given point x_0.

   When $f(x)$ becomes more and more positive (respectively more and more negative) as x approaches x_0, we say that $\lim_{x\to x_0} = +\infty$ (respectively $\lim_{x\to x_0} = -\infty$). Here the sign is very important. For instance, $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ (0^+ means that we are approaching 0 from the right), $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ (0^- means we are approaching 0 from the left) and then $\lim_{x\to 0} \frac{1}{x}$ does not exist, since we care about the sign of infinity.

   Now, consider the following graph and answer the following questions.



   (a) Does $\lim_{x\to a^-} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? Does it agree with $f(a)$? Does $\lim_{x\to a^+} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? Does it agree with $f(a)$? Does $\lim_{x\to a} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? **(1 pt)**

(b) Does $\lim_{x\to b^-} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? Does it agree with $f(b)$? Does $\lim_{x\to b^+} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? Does it agree with $f(b)$? Does $\lim_{x\to b} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? **(1 pt)**

(c) Does $\lim_{x\to c^-} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? Does it agree with $f(c)$? Does $\lim_{x\to c^+} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? Does it agree with $f(c)$? Does $\lim_{x\to c} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? **(1 pt)**

(d) Does $\lim_{x\to d^-} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? Does it agree with $f(d)$? Does $\lim_{x\to d^+} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? Does it agree with $f(d)$? Does $\lim_{x\to d} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? **(1 pt)**

(e) Does $\lim_{x\to e^-} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? Does it agree with $f(e)$? Does $\lim_{x\to e^+} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? Does it agree with $f(e)$? Does $\lim_{x\to e} f(x)$ exist? If so, is it finite, $+\infty$ or $-\infty$? **(1 pt)**

Solution:

(a) The left limit exists, it is finite, but it does not coincide with $f(a)$. The right limit exists, it is $-\infty$, it does not coicide with $f(a)$. Left and right limit do not agree, so the limit at a does not exist.

(b) The function is not defined at b, so there is no hope any limit could coincide with the value of f at b. The left limit exists, it is finite $+\infty$. The right limit exists, it is $-\infty$. Left and right limit do not agree, so the limit at b does not exist.

(c) The function is not defined at c, so there is no hope any limit could coincide with the value of f at c. The left limit exists, it is finite $-\infty$. The right limit exists, it is $-\infty$. Left and right limit agree, so the limit at c exists and it is $-\infty$.

(d) The left limit exists, it is finite, but it does not coincide with $f(d)$. The right limit exists, it is finite, but it does not coicide with $f(d)$. Left and right limit agree, so the limit at d exists and is finite.

(e) The left limit exists, it is finite, it dcoincides with $f(e)$. The right limit exists, it is finite, but it does not coicide with $f(e)$. Left and right limit do not agree, so the limit at e does not exist.

2. **(Using a graph for understanding the limit of a function)** Using the definitions in the previous problem (i.e. if the value of the function should go to $+\infty$ or $-\infty$, we say that the limit exists and is equal to $+\infty$ or $-\infty$ respectively), answer the following questions.

   (a) Sketch the graph of the function $\frac{\cos(x)}{|x|}$ (HINT: $\cos(x)$ and $|x|$ are both even functions). **(1 pt)**

(b) Does $\lim_{x \to 0^+} \dfrac{cos(x)}{|x|}$ exist? If so, what is its value? Look at the graph for providing an answer.**(1 pt)**

(c) Does $\lim_{x \to 0^-} \dfrac{cos(x)}{|x|}$ exist? If so, what is its value? Look at the graph for providing an answer.**(1 pt)**

(d) Does $\lim_{x \to 0} \dfrac{cos(x)}{|x|}$ exist? Explain why using part (b) and (c). If it exists, what is its value?**(1 pt)**

(e) Skecth the graph of the function $\frac{\cos(x)}{x}$ (HINT: $\cos(x)$ is an even function, while x

is an odd function). **(1 pt)**

(f) Does $\lim_{x \to 0^+} \dfrac{cos(x)}{x}$ exist? If so, what is its value? Look at the graph for providing an answer.**(1 pt)**

(g) Does $\lim_{x \to 0^-} \dfrac{cos(x)}{x}$ exist? If so, what is its value? Look at the graph for providing an answer.**(1 pt)**

(h) Does $\lim_{x\to 0} \dfrac{cos(x)}{x}$ exist? Explain why using part (f) and (g). If it exists, what is its value?**(1 pt)**

Solution:

(a)

(b) $\lim_{x\to 0^+} \frac{\cos(x)}{|x|} = +\infty$

(c) $\lim_{x\to 0^-} \frac{\cos(x)}{|x|} = +\infty$

(d) $\lim_{x\to 0} \frac{\cos(x)}{|x|} = +\infty$

(e)

(f) $\lim_{x\to 0^+} \frac{\cos(x)}{x} = +\infty$

(g) $\lim_{x\to 0^-} \frac{\cos(x)}{x} = -\infty$

(h) $\lim_{x\to 0} \frac{\cos(x)}{x}$ does not exist

3. **(The squeezing theorem)(See page 110)** In this exercise, the squeeze theorem is helpful. (The Squeeze Theorem says if $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$ where L can be $+\infty$ or $-\infty$. The conclusion holds also for one-sided limit.) In the last lab we encountered the function $\sin\left(\frac{\pi}{x}\right)$ and we saw it does not have limit as x approaches 0. Now we consider the function $x \sin\left(\frac{\pi}{x}\right)$, which is defined for $x \neq 0$. Although it is not defined in 0, it makes perfectly sense to see whether $\lim_{x \to 0} x \sin\left(\frac{\pi}{x}\right)$ exists, since for answering this question we care about what happens very close to 0, but we neglect the point 0.

   Notice that $-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$ where $\sin\left(\frac{\pi}{x}\right)$ is defined (i.e. away from 0).

   (a) Use the hint to find two functions $f(x)$ and $h(x)$ so that $f(x) \leq x \sin\left(\frac{\pi}{x}\right) \leq h(x)$ when x is near 0 and which can be used to apply the squeezing theorem (so we want that $\lim_{x \to 0} f(x)$ exists, $\lim_{x \to 0} h(x)$ exists and $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x)$). **(1 pt)**

   (b) Sketch the graph of $f(x)$, $g(x)$ and $x \sin\left(\frac{\pi}{x}\right)$, all in one Cartesian plane. **(1 pt)**

(c) Using part (a) and the squeezing theorem, show that $\lim_{x\to 0} x \sin\left(\frac{\pi}{x}\right)$ exists and provide the actual value. State explicitly which are the functions $f(x)$, $g(x)$ and $h(x)$ appearing in the squeezing theorem above in this precise case. **(1 pt)**

Solution:

(a) $f(x) = -|x|$, $g(x) = |x|$

(b)

(c) $\lim_{x\to 0} -x = \lim_{x\to 0} x = 0$, hence by the squeezing theorem $\lim_{x\to 0} x \sin\left(\frac{\pi}{x}\right) = 0$.

4. **(Some computations)(See Section 2.3)**

(a) Does $\lim_{x\to 3} \dfrac{x^3-27}{x-3}$ exist? If so, what is its value? Hint: try to factor the numerator and use the "direct substitution property" at page 107 **(1 pt)**

(b) Using part (a) and the properties at page 104, calculate $\lim_{x\to 3}\left(e^x + \dfrac{x^3-27}{x-3}\right)$. **(1 pt)**

(c) Does $\lim_{x \to -3} \dfrac{x^3 + 27}{x + 3}$ exist? If so, what is its value? Hint: try to factor the numerator and use the "direct substitution property" at page 107 **(1 pt)**

(d) Using part (c) and the properties at page 104, calculate $\lim_{x \to -3} \left(\dfrac{1}{x - 4} \cdot \dfrac{x^3 + 27}{x + 3} \right)$. **(1 pt)**

(e) Does $\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}$ exist? If so, what is its value? Hint: try to look at example 6 in Section 2.3. **(1 pt)**

(f) Recall that the Gauss floor function $[x]$ means the greatest integer that is less than or equal to x. Now, consider the function $[\sin(x)]$.

- Evaluate the function $[\sin(x)]$ at the points 0, $\pi/2$ and $-\pi/2$. After having evaluated these values, calculate the limits $\lim_{x\to 0^-}[\sin(x)]$, $\lim_{x\to 0^+}[\sin(x)]$, $\lim_{x\to \frac{\pi}{2}^-}[\sin(x)]$, $\lim_{x\to \frac{\pi}{2}^+}[\sin(x)]$, $\lim_{x\to -\frac{\pi}{2}^-}[\sin(x)]$ and $\lim_{x\to -\frac{\pi}{2}^+}[\sin(x)]$. **(2 pts)**
- Bearing in mind what you calculated in the previous part, sketch the graph of $[\sin(x)]$. Recall that a function f is said to be continuous at a point x_0 if $\lim_{x\to x_0} f(x)$ exists, it is finite and we have $f(x_0) = \lim_{x\to x_0} f(x)$. As you can see from the graph you sketched, the function $[\sin(x)]$ is continuous at most of the points. At what points is it not continuous? (Hint: remember that $\sin(x)$ is a periodic function with period 2π) **(2 pts)**

Solution:

(a) $x^3-27=(x-3)(x^2+3x+9)$, so $(x^3-27)/(x-3)=x^2+3x+9$. This equality holds away from 3, since there $(x^3-27)/(x-3)$ is not defined; anyway we do not need the value at the point 3 to evaluate $\lim_{x\to 3}$. So we have $\lim_{x\to 3}(x^3-27)/(x-3)=\lim_{x\to 3} x^2+3x+9$. Then by the direct substitution property $\lim_{x\to 3} x^2+3x+9=3^2+3\cdot 3+9=27$.

(b) Both $\lim_{x\to 3}(x^3-27)/(x-3)$ and $\lim_{x\to 3} e^x$ exist, so we have $\lim_{x\to 3}(e^x+(x^3-27)/(x-3))=\lim_{x\to 3} e^x+\lim_{x\to 3}(x^3-27)/(x-3)=e^3+27$.

(c) $x^3+27=(x+3)(x^2-3x+9)$, so $(x^3+27)/(x+3)=x^2-3x+9$. This equality holds away from -3, since there $(x^3+27)/(x+3)$ is not defined; anyway we do not need the value at the point -3 to evaluate $\lim_{x\to -3}$. So we have $\lim_{x\to -3}(x^3+27)/(x+3)=\lim_{x\to -3} x^2-3x+9$. Then by the direct substitution property $\lim_{x\to -3} x^2-3x+9=(-3)^2+(-3)\cdot(-3)+9=27$.

(d) Both $\lim_{x\to -3}(x^3+27)/(x-3)$ and $\lim_{x\to -3}(x-4)$ exist. Also, $\lim_{x\to -3}(x-4)=-7$ is not zero. So we have $\lim_{x\to -3}\left(\frac{1}{x-4}\cdot\frac{x^3+27}{x+3}\right)=(\lim_{x\to -3}(x^3+27)/(x+3))/(\lim_{x\to -3}(x-4))=27/(-7)=-\frac{27}{7}$.

(e) Since we are around 16, we can take $\sqrt{x}$, so we can write

$$\frac{\sqrt{x}-4}{x-16}=\frac{\sqrt{x}-4}{(\sqrt{x}-4)(\sqrt{x}+4)}$$

The denominator is 0 just in 16, but since we are interested in the limit at 16 we can neglect the value of the function at 16. So we can simplify and get

$$\frac{\sqrt{x}-4}{x-16}=\sqrt{x}+4$$

near 16 and so we have $\lim_{x\to 16}\frac{\sqrt{x}-4}{x-16}=\lim_{x\to 16}\sqrt{x}+4=8$.

(f) $[\sin(0)]=0$, $\left[\sin\left(\frac{\pi}{2}\right)\right]=1$, $\left[\sin\left(-\frac{\pi}{2}\right)\right]=-1$. Then we have $\lim_{x\to 0^+}[\sin(x)]=0$, $\lim_{x\to 0^-}[\sin(x)]=-1$, $\lim_{x\to \frac{\pi}{2}^+}[\sin(x)]=0$, $\lim_{x\to \frac{\pi}{2}^-}[\sin(x)]=0$, $\lim_{x\to -\frac{\pi}{2}^+}[\sin(x)]=-1$, $\lim_{x\to -\frac{\pi}{2}^-}[\sin(x)]=-1$.

We can see that at 0 and at $\frac{\pi}{2}$ we have troubles. We can see, analogously as at 0, that we have discontinuity also at π. Now we can use the periodicity of $\sin(x)$ to conclude that the function is discontinuous at the points having form either $k\pi$, where k is any integer number, or $\frac{\pi}{2}+2k\pi$, where k is any integer number.