Math 1310 Lab 10. (Sections 5.1-5.3)

Name/Unid: ________________________________ Lab section: ____

1. **(Properties of the integral)**

   Use the properties of the integral in section 5.2 for answering the following question.

   (a) Knowing that

   $$\int_0^2 2f(x) + 3dx = 8, \quad \int_0^2 f(x) - g(x)dx = -1,$$

   calculate $\int_0^2 g(x)dx$. **(2 pts)**

   (b) We know that that $\int_0^1 3x^2 - f(x)dx = 0$, $f(x)$ is constant for $x \geq 1$ and $\int_0^2 f(x)dx = 0$. Determine $f(1)$. **(2 pts)**

Solution:

(a) We can rewrite the fist integral as $2\int_0^2 f(x)dx + \int_0^2 3dx = 8$, thus we get $\int_0^2 f(x)dx = 1$. We can rewrite the second integral as $\int_0^2 f(x) - \int_2^2 g(x)dx = -1$, which gives $\int_0^2 g(x) = 2$.

(b) x^3 is an antiderivative of $3x^2$, so $\int_0^1 3x^2dx = [x^3]_0^1 = 1$. Thus $\int_0^1 f(x)dx = 1$. If we split the second integral given and substitute 1 for $\int_0^1 f(x)dx$, we get $1 + \int_1^2 f(x)dx = 0$. Hence $\int_1^2 f(x)dx = -1$. But it has to be the integral of a constant, say c, so we get $(2-1)c = -1$. It gives us $c = -1$ and thus $f(1) = -1$.

2. **(A little theorem)** Assume $f(x)$ is a continuous function and that $f(x) \geq 0$ for all real numbers. If $\int_0^1 f(x)dx = 0$, we have that $f(x) = 0$ in the interval $[0.1]$. We will show it in a few steps.

   (a) Pick c in the interval $[0, 1]$. If $f(c) > 0$, what does continuity tell us about $f(x)$ for x very close to c? **(1 pt)**

   (b) Consider c as above and assume $f(c) = d > 0$. An argument analogous to part (a) tells us that $f(x) \geq \frac{d}{2}$ for $c - \delta \leq x \leq c + \delta$, where δ is a small positive number. Use "Comparison Properties of the Integral" for arguing that $\int_{c-\delta}^{c} f(x)dx > 0$ and $\int_{c}^{c+\delta} f(x)dx > 0$. **(3 pts)**

(c) Use part (a) and (b), "Properties of the Integral" and "Comparison properties of the Integral" to argue that if $f(x) \geq 0$ and $f(c) > 0$ for some c in the interval $[0, 1]$, then $\int_0^1 f(x)dx > 0$. Be careful in breaking the integral: if $c = 0$ or $c = 1$ you will need just one of the two integrals in (b), while if $0 < c < 1$ we choose δ very small so that $0 < c - \delta < c < c + \delta < 1$ and we use both integrals in (b). **(3 pts)**

This little theorem applies also in the following case: $f(x)$ and $g(x)$ are continuous and $f(x) \geq g(x)$. If $\int_0^1 f(x)dx = \int_0^1 g(x)dx$, then $f(x) = g(x)$ for $0 \leq x \leq 1$. Indeed, we can apply the theorem in the exercise to the function $f(x) - g(x)$.

Solution:

(a) Continuity tells us that $f(x)$ has to be positive for x close to c.

(b) It is given that $f(x) \geq \frac{d}{2}$ in the interval $[c-\delta, c+\delta]$. Thus we have $\int_c^{c+\delta} f(x)dx \geq \int_c^{c+\delta} \frac{d}{2}dx = \delta\frac{d}{2} > 0$. We argue analogously for the other integral.

(c) If $c = 0$, we break the integral as $\int_0^{\delta} f(x)dx + \int_{\delta}^{1} f(x)dx$. Part (b) tells that the first integral is positive and comparison $f(x) \geq 0$ tells that the second integral is non-negative; hence their sum is positive. If $c = 1$ we break the integral as $\int_0^{1-\delta} f(x)dx + \int_{1-\delta}^{1} f(x)dx$. If $0 < c < 1$ we break the integral as $\int_0^{c-\delta} f(x)dx + \int_{c-\delta}^{c+\delta} f(x)dx + \int_{c+\delta}^{1} f(x)dx$.

3. **(Integrals as areas)** A car is traveling on a highway. For the first hour it travels at constant speed $90km/h$. Then the driver takes a break for lunch for fifteen minutes. After the break, the car travels at $80km/h$ for half an hour.

   (a) Sketch the graph of the velocity of the car. **(1 pt)**

   (b) How far has the car traveled after 1 hour and ten minutes? And after the whole 1 hour and 45 minutes? **(2 pts)**

   (c) The driver has to travel $160km$ in 2 hours. The data given tells us that, when 15 minutes are left, the driver is driving at $80km/h$. Assume that the car accelerates with constant acceleration from then on. Determine the acceleration that would guarantee the driver to cover exactly $160km$ in the 2 hours he has. **(3 pts)**

Solution:

(a) After 1 hour and ten it has traveled 90 kilometers. After 1 hour and 45 minutes it has traveled 130 kilometers.

(b) From part (b) we know that in 15 minutes it has to cover 30 kilometers. If we interpret the distance covered in the last 15 minutes as the area of a trapezoid with height $\frac{1}{4}$, one base 80 and the other base to determine, we get $\frac{80+B}{8} = 30$, which gives us $B = 160$. Then the slope is $\frac{160-80}{0.25} = 320km/h^2$.

4. **(Some computation)** Use the properties in sections 5.1-5.3 for calculating the following integrals and limits.

   (a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$ **(2 pts)**

   (b) $\int_{\frac{1}{n}}^{1} \frac{1}{x} dx$ where n is a positive integer **(2 pts)**

   (c) $\int_{1}^{n} \frac{1}{x} dx$ where n is a positive integer **(2 pts)**

   (d) $\int_{a}^{0} e^x dx$ where $a < 0$ **(2 pts)**

Solution:

(a) We recognize the definition of integral for $\int_0^1 x^3 dx = \frac{1}{4}$.

(b) An antiderivative is $\ln(x)$, thus we get $\ln(n) - \ln(1) = \ln(n)$.

(c) An antiderivative is $\ln(n)$, thus we get $\ln(1) - \ln\left(\frac{1}{n}\right) = \ln(n)$.

(d) An antiderivative is e^w, thus we get $e^0 - e^a = 1 - e^a$.