1. (The differential of a circular cone)

Here the angle $\alpha$ is fixed. Observe the radius of the smaller circle, $r$, equals $h \cdot \tan \alpha$, and the radius of the larger circle equals $(h + \Delta h) \cdot \tan \alpha$.

**Question 1.** What is the volume $V$ of the smaller circular cone (with radius $r$), in terms of $\alpha, h$? (1 pt)
**Question 2.** What is the volume $\Delta V$ of the flat cone bounded by the two circles, in terms of $\alpha, h, \Delta h$? (Hint: this flat cone is the different of two circular cones.) \(1 \text{ pt}\)

As you can see in the figure, the volume of the flat cone $\Delta V$ is smaller than the volume of the cylinder with radius $PC$, but greater than the volume of the cylinder with radius $OB$. This implies $\pi(h \tan \alpha)^2 \Delta h \leq \Delta V \leq \pi((h + \Delta h) \tan \alpha)^2 \Delta h$. But anyway, when $\Delta h$ is small, the volume of these cylinders to the flat cone is “almost” the same. Do the next problem.

**Question 3.** Show that

$$\lim_{\Delta h \to 0} \frac{\Delta V}{\pi(h \tan \alpha)^2 \Delta h} \to 1.$$ 

(Hint: to play with this limit, think of $h, \alpha$ as constants, and $\Delta h$ as a variable.) \(2 \text{ pts}\)

The above limit suggests that $dV \approx \pi(h \tan \alpha)^2 \, dh$. Here $d$ means very small difference ($\Delta$).
**Question 4.** Differentiate $V$ in Question 1. with respect to variable $h$ (Keep $\alpha$ as a constant). Use this derivative to write a formula for the differential $dV$. (Compare your answer with the comment in the very end of the previous page.) (2 pts)

**Question 5.** Use the formula you have derived for $dV$ to do the following problem: given a circular cone of radius 5 cm and height 7 cm. If we increase the height by 0.01 cm to get a larger circular cone, then “roughly” how much volume is increased? (Hint: What is $\tan \alpha$?) (You can include $\pi$ in your answer, or set it equal to 3.14.) (2 pts)

**Solution:**

A1. $\frac{1}{3} \pi r^2 h = \frac{1}{3} \pi h^3 \tan^2 \alpha$.

A2. $\frac{1}{3} \pi (h + \Delta h)^3 \tan^2 \alpha - \frac{1}{3} \pi h^3 \tan^2 \alpha = \frac{1}{3} \pi \tan^2 \alpha (3h^2 \Delta h + 3h(\Delta h)^2 + (\Delta h)^3)$.

A3. Take the inequality given, divide everything by $\pi (h \tan \alpha)^2 \Delta h$ and apply the squeezing theorem.

A4. $V'(h) = h^2 \pi \tan^2 \alpha$, which implies $dV = h^2 \pi \tan^2 \alpha \, dh$.

A5. Plug in these values, we have $dV = 7^2 \cdot \pi \cdot (\frac{5}{7})^2 \cdot 0.01 \approx 0.785$ (cm$^3$)
2. (Related rates: circular motion) (See also Lab 2, problem 1.)

A particle is moving on a unit circle $x^2 + y^2 = 1$. The $x, y$ coordinates are functions of time $t$, i.e., $x = x(t), y = y(t)$. We’re interested in how $\frac{dx}{dt}$ relates to $\frac{dy}{dt}$, provided that $x(t), y(t)$ are smooth functions.

**Question 1.** Assume that at some instant, the position of the moving particle is $x = 1/2, y = \sqrt{3}/2$. If we know $\frac{dy}{dt} = 2$ at that point, then what is $\frac{dx}{dt}$? (Hint: differentiate $x^2 + y^2 = 1$ with respect to $t$.) (2 pts)

**Question 2.** One possible parameterization of the circular motion is given by $x = \cos(4t), y = \sin(4t)$. If we let $t = \frac{x}{12}$, then what is $x$? What is $y$? What is $\frac{dy}{dt}$ (evaluated at $t = \frac{x}{12}$)? What is $\frac{dx}{dt}$ (evaluated at $t = \frac{x}{12}$)? (2 pts)
Question 3. Another parameterization of the circular motion is given by $x = \cos(t^2 + \frac{\pi}{3} - 4)$, $y = \sin(t^2 + \frac{\pi}{3} - 4)$. If we let $t = 2$, then what is $x$? What is $y$? What is $\frac{dy}{dt}$ (evaluated at $t = 2$)? What is $\frac{dx}{dt}$ (evaluated at $t = 2$)? (2 pts)

These examples show us that, when we calculate the related rates, the answer remains the same no matter how we parameterize the curve.

Solution: A1. $(2x) \cdot \frac{dx}{dt} + (2y) \cdot \frac{dy}{dt} = 0$. Therefore, when $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$, $\frac{dy}{dt} = 2$, then $\frac{dx}{dt} = -2\sqrt{3}$.

A2. $\cos(4t) \bigg|_{t=\frac{\pi}{12}} = \frac{1}{2}$, $\sin(4t) \bigg|_{t=\frac{\pi}{12}} = \frac{\sqrt{3}}{2}$, $\frac{dy}{dt} = 4 \cos(4t) \bigg|_{t=\frac{\pi}{12}} = 2$, $\frac{dx}{dt} = -4 \sin(4t) \bigg|_{t=\frac{\pi}{12}} = -2\sqrt{3}$.

A3. $\cos(t^2 + \frac{\pi}{3} - 4) \bigg|_{t=2} = \frac{1}{2}$, $\sin(t^2 + \frac{\pi}{3} - 4) \bigg|_{t=2} = \frac{\sqrt{3}}{2}$, $\frac{dy}{dt} = 2t \cos(t^2 + \frac{\pi}{3} - 4) \bigg|_{t=2} = 2$, $\frac{dx}{dt} = -4 \sin(t^2 + \frac{\pi}{3} - 4) \bigg|_{t=2} = -2\sqrt{3}$.
3. (Critical values) (See Sections 4.2 and 4.3)

A cubic function is a polynomial of degree 3, i.e. a polynomial of the form \( p(x) = ax^3 + bx^2 + cx + d \) where \( a \neq 0 \).

(a) Show by means of examples that a cubic function can have zero, one or two critical numbers. For each of the three examples sketch a graph on a cartesian plane. (2 pts)

(b) How many local extreme values can a cubic function have? Provide an example for each possible case and provide an explanation for not possible cases. (2 pts)
(c) Provide an example of a cubic function having one critical number but no extreme values. (1 pt)

(d) Show that a cubic function can not have two critical numbers and no extreme values at the same time. (2 pts)

Solution:
(a) In order, $x^3 + x$, $x^3$, $x^3 - x$.

(b) Two: $x^3 - x$, zero $x^3$. A cubic cannot have just one local extreme value. For a cubic function $f(x)$ either $\lim_{x \to \pm\infty} f(x) = +\infty$ or $\lim_{x \to \pm\infty} f(x) = -\infty$. On the other hand a function $f(x)$ attaining just one local extreme value either first increases and then decreases or first decreases and then increases: in both cases $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ cannot be infinite of the same sign. Hence a cubic function cannot have just one local extreme value.

(c) $x^3$

(d) If a cubic function has two critical numbers, it means that the first derivative is a second degree polynomial with two distinct roots. On the other hand for having no extreme values we want the first derivative to be always either nonnegative or nonpositive. This is not possible for a second degree polynomial with two distinct roots.
4. (More critical values and graphs) (See Sections 4.2 and 4.3)

Find the domain of the following functions. Then calculate their first and second derivative. After that sketch a graph for each of those (NOTE: it has to be consistent with the information the first and second derivative provide you!). After that, for each of them find the absolute maximum and absolute minimum in the given interval.

(a) \( x^3 - x \), for the extreme values consider \([0, 3]\) (2 pts);

(b) \( x - \ln x \), for the extreme values consider \((0, +\infty)\) (2 pts).
Solution:

(a) The domain is all the real numbers. The first derivative is $3x^2 - 1$ and the second derivative is $6x$. The critical values are $\frac{\sqrt{3}}{3}$ and $-\frac{\sqrt{3}}{3}$. The function is concave for $x < 0$ and convex for $x > 0$. The only critical value in $[0, 3]$ is $\frac{\sqrt{3}}{3}$ and the second derivative test says it is a local minimum. At the extrema of the interval the function takes values 0 (at 0) and 24 (at 3). Hence the function on $[0, 3]$ has maximum at 3 (24) and minimum at $\frac{\sqrt{3}}{3} (-2\sqrt{3})$.

(b) The domain is $(0, +\infty)$ because of the presence of the logarithm. The first derivative is $1 - \frac{1}{x}$ (and hence the only critical number is 1), while the second derivative is $\frac{1}{x^2}$, so the function is always convex. Since the function is always convex and we have just one critical number, it is the global minimum. On the other hand, there is no maximum point, since $\lim_{x \to 0} f(x) = \lim_{x \to +\infty} f(x) = +\infty$. 