Math 1310 Lab 7.
Name/Unid: _______________________________ Lab section: ____

1. (Derivatives of inverse functions)
 Determine the domain of each of the following functions. Then, using the properties in sections 3.4, 3.6 and 3.7, calculate the derivatives of the following functions. Remember that $\arccos(x) = \cos^{-1}(x)$, $\arcsin(x) = \sin^{-1}(x)$, $\arctan(x) = \tan^{-1}(x)$.

(a) $\sin(x^5)$ (2 pts)

(b) $\sin^5(x)$ (2 pts)

(c) $\arcsin(\arctan(x))$ (2 pts)
(d) ln(arctan(x)) (2 pts)

Solution:

(a) The domain is all the real numbers. $\sin(x^5) = f(g(x))$ where $f(x) = \sin(x)$ and $g(x) = x^5$. Hence the chain rule tells us that the derivative is $\cos(x^5) \cdot 5x^4$.

(b) The domain is all the real numbers. $\sin^5(x) = f(g(x))$ where $f(x) = x^5$ and $g(x) = \sin(x)$. Hence the chain rule tells us that the derivative is $5\sin^4(x) \cdot \cos(x)$.

(c) The domain is $[-\tan(-1), \tan(1)]$. Then the chain rule gives us

$$\frac{1}{\sqrt{1-\arctan^2(x)}} \frac{1}{1+x^2}$$

(d) The domain is $(0, +\infty)$. The chain rule gives us

$$\frac{1}{\arctan(x)} \frac{1}{1+x^2}.$$
2. (Some more advanced computations) Using the properties in sections 3.4, 3.6 and 3.7, answer the following questions.  

(a) Compute the limit \( \lim_{x \to 0} \frac{\sin(5x)}{2x} \) (2 pts) 

(b) Suppose \( f \) is an injective (i.e. one-to-one) differentiable function and its inverse function \( f^{-1} \) is also differentiable. Use implicit differentiation to show that 

\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}
\]

provided that the denominator is not 0. Furthermore, assuming that \( f(5) = 3 \) and \( f'(5) = 8 \), calculate \((f^{-1})'(3)\). (3 pts)
(c) Using the result in given above in part 2.(b) and the relation between the functions $e^x$ and $\ln(x)$, prove the following equality

$$\lim_{x \to 0} \frac{\ln(x + 1)}{x} = 1$$

Hint: what derivative can you recognize in the above equality? (2 pts)

(d) Determine the domain of the function $f(x) = \frac{1+x^2}{\sqrt{1-x^2}}$. Using logarithmic differentiation, find $f'(x)$. Use $f'(x)$ to determine the tangent line to the graph of $f(x)$ at the point $(0,1)$. (3 pts)

Solution:

(a) The derivative of the sine is the cosine, and the sine in 0 is 0. So $\lim_{x \to 0} \frac{\sin(x)}{x} = \cos(0) = 1$. Hence the limit is $\frac{5}{2}$.

(b) We have that $f(f^{-1}(x)) = x$, so if we take the derivative both sides with respect to $x$, according to the chain rule we get $f'(f^{-1}(x))(f^{-1})'(x) = 1$. The assumption is that $f'(f^{-1}(x)) \neq 0$, so we can divide and get the claimed identity. In the particular case as above we get $(f^{-1})'(3) = \frac{1}{f'(5)} = \frac{1}{8}$.

(c) The left hand side gives the derivative of the function $\ln(z)$ at $z = 1$. Also, $\ln(1) = 0$ and logarithm and exponential are inverse to each other. Hence by the formula in 2.(b) we have

$$\lim_{x \to 0} \frac{\ln(x + 1)}{x} = \ln'(1)$$

$$= \frac{1}{\exp'(0)}$$

$$= 1$$

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(d) The domain is $(-1, 1)$, since we want nonnegative numbers under the square root and also the denominator cannot be 0. Taking the natural logarithm both sides in $y = \frac{1+x^2}{\sqrt{1-x^2}}$ we get

$$\ln(y) = \ln(\frac{1+x^2}{\sqrt{1-x^2}})$$
$$= \ln(1+x^2) - \ln(\sqrt{1-x^2})$$
$$= \ln(1+x^2) - \frac{1}{2} \ln(1-x^2).$$

Now we take the derivative of this identity and get

$$\frac{y'}{y} = \frac{2x}{1+x^2} + \frac{x}{1-x^2};$$

(1)

since $y = \frac{1+x^2}{\sqrt{1-x^2}}$, we get

$$y' = \frac{1+x^2}{\sqrt{1-x^2}}(\frac{2x}{1+x^2} + \frac{x}{1-x^2}).$$

Plugging in 0 in $y'$ we get that $f'(0) = 0$, so the tangent line we are looking for is $y = 1$. 

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3. (Ohm’s law)

In the theory of electrical circuits, Ohm’s law describes the relationship between the voltage $V$ across a resistor, the electrical current $I$ passing through the resistor, and a quantity $R$ known as the resistance. The law can be written as follows:

$$V = IR$$

Usually voltage is measured in volts (symbol $V$), current is measured in amperes (symbol $A$), and resistance is measured in ohms (symbol $\Omega$), where 1 ohm = 1 volt/ampere. In a circuit with variable resistance, the quantities $V$, $I$, and $R$ might all depend on time.

(a) Take the derivative of Ohm’s law to find an equation relating $\frac{dV}{dt}$, $\frac{dI}{dt}$, and $\frac{dR}{dt}$. (1 pt)

(b) Suppose that the current is increasing at a rate of 0.3 A/s, while the resistance is holding steady at 4 $\Omega$. How quickly is the voltage across the resistor increasing? (2 pts)

Now suppose that the voltage across the resistor is held constant at 20 V, while the resistance is steadily increased at a rate of 0.4 $\Omega$/s.

(c) What is the current through the resistor when the resistance reaches 10 $\Omega$? (2 pts)

(d) At what rate is the current changing at that time? Is the current increasing or decreasing? (2 pts)

**Solution:**

(a) Using the product rule we get

$$\frac{dV}{dt} = \frac{dR}{dt}I + \frac{dI}{dt}R.$$  

(b) Using the above formula and knowing that $R = 4 \Omega$, and $\frac{dI}{dt} = 0.3 A/s$, we get $\frac{dV}{dt} = 1.2 V/s$.

(c) We know that $IR = V$ and that $V = 20 V$, so when $R = 10 \Omega$ we get $I = 2 A$.

(d) We know that $IR = 20 V$ and that $\frac{dV}{dt} = 0$, hence $0 = \frac{dI}{dt}R + \frac{dR}{dt}I$. When $R = 10 \Omega$, then $I = 2 A$. So we get $0 = (0.4 \Omega/s)(2 A) + \frac{dI}{dt}10 \Omega$, so we get $\frac{dI}{dt} = -0.08 A/s$. Hence $I$ is decreasing.