AN INTRODUCTION TO THE MODULI PROBLEM FOR VARIETIES OF GENERAL TYPE
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ABSTRACT. These notes are a short introduction to the moduli problem for algebraic varieties. Starting from the well known case of curves, we analyze the obstructions to construct a fine moduli, and deduce what is the best we can hope for. Once expectations are established, by means of examples we point out the right generalization to higher dimensions in order to have a well behaved moduli problem.

1. THE MODULI PROBLEM

In mathematics, one of the main problems is the one of the classification. Given a class of objects, one would like to find an optimal "set" of representatives, which hopefully has some functorial properties.

In algebraic geometry, the above problem loosely translates into finding a nice parameter space (hopefully something close to a variety) and a universal family (what would assure the right functoriality) for a suitable family of varieties, sheaves, etc.. This concept is better illustrated by means of example.

Example 1.1 (Hilbert scheme, I)**.** Assume k is an algebraically closed field, and X is a projective variety over k. A natural question is to find a parameter space for the set

$$Hilb(X) = \{\text{closed subschemes of } X\}.$$

One can show that there is a k-scheme $\mathrm{Hilb}(X)$, the Hilbert scheme of X, that serves this purpose. For each Hilbert polynomial p[1], there are an irreducible component $\mathrm{Hilb}_p(X) \subset \mathrm{Hilb}(X)$ and a universal family $\mathrm{Univ}_p(X) \subset \mathrm{Hilb}_p(X) \times X$ such that:

- $\mathrm{Hilb}_p(X)$ is projective;
- $\pi : \mathrm{Univ}_p(X) \to \mathrm{Hilb}_p(X)$ is flat;
- for any $Y \subset X$ with Hilbert polynomial p, there exist a unique $[Y] \in \mathrm{Hilb}_p(X)$ such that $\pi^{-1}([Y]) = Y$;
- for every scheme T, and every closed subscheme $Z \subset T \times X$ flat over X with Hilbert polynomial p, there exist a unique $f : T \to \mathrm{Hilb}_p(X)$ such that $Z = T \times_{\mathrm{Hilb}_p(X)} \mathrm{Univ}_p(X)$.

Therefore, the hope of an algebraic geometer is that for a given class of objects $\mathbf{V}$ (e.g. isomorphism classes of curves of genus g, vector bundles on a projective variety X, etc.), there exist a projective scheme $\mathrm{Moduli}_{\mathbf{V}}$, and a flat and proper morphism $u : \mathrm{Univ}_{\mathbf{V}} \to \mathrm{Moduli}_{\mathbf{V}}$ such that any flat family of objects in $\mathbf{V}$ is obtained from u by base change. In this case, $\mathrm{Moduli}_{\mathbf{V}}$ is called *fine moduli space*, and $\mathrm{Univ}_{\mathbf{V}}$ is the corresponding *universal*

[1]This is a numerical invariant that captures dimension, and degree with respect to a previously fixed ample class on X.

family. It is worth pointing out that flatness of the families considered is a necessary (yet not intuitive) condition to put in order to have a well behaved moduli problem. Therefore, from now on, the families of varieties will assumed to be flat, unless otherwise stated.

On the other hand, this ideal program does not go through in most cases, and the Hilbert scheme represents a very rare full success. In certain cases the scheme $\mathrm{Moduli}_{\mathbf{V}}$ may have bad properties (e.g. not separated or proper, not of finite type). More importantly, in most cases there is no universal family $\mathrm{Univ}_{\mathbf{V}}$; in such case, $\mathrm{Moduli}_{\mathbf{V}}$ is called *coarse moduli space*. Sometimes, there is even no parameter space. Thus, the perspective on the problem should be shifted.

Example 1.2 (Hilbert scheme, II)**.** We actually realize that we can rephrase the construction in Example 1.1 in the language of functors. We will write

$$Hilb_X(-) : \mathfrak{Sch}_k \to \mathfrak{Sets}$$

for the contravariant functor that to every k-scheme T associates the set

$$\{\text{subschemes } \; Z \subset T \times X \text{ that are flat and proper over } \; T\}.$$

Morphisms $f : S \to T$ at level of schemes translate by base change to maps between sets of flat and proper families as follows:

$$\begin{aligned} Hilb_X(f) : Hilb_X(T) &\to Hilb_X(S) \\ Z &\mapsto Z \times_T S \end{aligned}$$

In this setting, the existence of a fine moduli space is equivalent to the *representability* of the given functor. Indeed, Example 1.1 guarantees us that $Hilb_X(-)$ is represented by $\mathrm{Hom}_{\mathfrak{Sch}_k}(-, \mathrm{Hilb}(X))$, where $\mathrm{Hilb}(X) = \cup_p \mathrm{Hilb}_p(X)$. Under this natural correspondence, the identity morphism $\mathrm{id}_{\mathrm{Hilb}(X)}$ provides us with the universal family $\mathrm{Univ}(X) = \cup_p \mathrm{Univ}_p(X)$.

For more details about the material of this section, see [HK10][Chapters 11, 12] and [Kol17][Chapter 1].

2. Properties of moduli functors

As mentioned in the previous section, it is very rare for a moduli functor to be representable. Since there may not even be a coarse moduli space $\mathrm{Moduli}_{\mathbf{V}}$ for our problem, we should find a new way to characterize certain properties. For instance, we would like to talk about separatedness, properness, and boundedness for our moduli problem. Following what happens in the case we have a fine moduli space satisfying the property $\mathcal{P}$, we will make sense of what means for a functor to satisfy $\mathcal{P}$. Following the notation introduced for the Hilbert functor, we will denote by

$$Moduli_{\mathbf{V}}(-) : \mathfrak{Sch}_k \to \mathfrak{Sets}$$

the contravariant functor that to every k-scheme T associates the set

$$\{\text{flat families } \; \mathcal{X} \to T \text{ such that every fiber is in } \; \mathbf{V}, \text{ modulo isomorphism over } \; T\}.$$

2.1. **Separatedness.** Assume we have a moduli problem for a class of varieties $\mathbf{V}$, and assume we have a fine moduli space $\mathrm{Moduli}_{\mathbf{V}}$. Also, consider a pointed curve (B, b). If $\mathrm{Moduli}_{\mathbf{V}}$ is a separated scheme, then there exist at most one diagonal arrow completing the following commutative diagram

$$\begin{array}{ccc} B \setminus \{b\} & \longrightarrow & \mathrm{Moduli}_{\mathbf{V}} \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & \operatorname{Spec} k \end{array}$$

Since we are assuming $\mathrm{Moduli}_{\mathbf{V}}$ is a fine moduli space, the above diagram is equivalent to saying that any flat family $\mathcal{X} \in \mathit{Moduli}_{\mathbf{V}}(B \setminus \{b\})$ has at most one flat extension $\mathcal{X}' \in \mathit{Moduli}_{\mathbf{V}}(B)$ through b. This suggests what the right definition of separated (moduli) functor should be.

Definition 2.1. A moduli functor $\mathit{Moduli}_{\mathbf{V}}$ is separated if for every smooth curve B and every open subset $B^0 \subset B$, any flat family $\mathcal{X} \in \mathit{Moduli}_{\mathbf{V}}(B^0)$ has at most one extension $\mathcal{X}' \in \mathit{Moduli}_{\mathbf{V}}(B)$.

Remark 2.2. A more algebraic definition using DVRs could be given [HK10][Chapter 13.D], but the above one, taken from [Kol17][pp. 12-13], is equivalent for our purposes.

In the modern theory of moduli spaces, one of the main goals is to generalize the successful construction of a coarse moduli space for curves of genus $g \geq 2$ to higher dimensions. The correct perspective on this is to consider smooth curves of genus $g \geq 2$ as curves of general type. As hinted in the following of these notes, and nicely explained in [Kol17] and [HK10], the presence of a preferred "polarization" (ω_X is just big so far) is a key point.

Example 2.3. Now, let $\mathbf{V} = \{$smooth projective surfaces of general type$\}$. Following [Kol17][pp. 15-16], we will show that $\mathit{Moduli}_{\mathbf{V}}$ is not separated. Consider a smooth family of projective surfaces $f : X \to B$ over a smooth affine curve B. Also assume there are three sections $C_1, C_2, C_3 \subset X$ of f that meet pairwise transverally at a point x_b mapping to $b \in B$. A way to obtain such a kind of family is to start with an arbitrary family, consider a curve $C \subset X$ that deforms, and that dominate the base B; the base change $X \times_B C \to C$ should provide the desired configuration. Also, since B is affine, up to shrinking the base, we may assume that x_b is the only point of intersection among the three sections.

Now, consider $X^1 = Bl_{C_1} Bl_{C_2} Bl_{C_3} X$ and $X^2 = Bl_{C_1} Bl_{C_3} Bl_{C_2} X$, where we are abusing notation for the strict transform of the C_i's on the previous blow-up. Since X^1 and X^2 are obtained blowing up sections of a smooth family, then $X^i \to B$ are still smooth families. Also, since over $B \setminus \{b\}$ the three sections are disjoint, X^1 and X^2 are isomorphic over $B \setminus \{b\}$.

As a general fact, we have an isomprhism between the central fibers of the first two blow-ups $(Bl_{C_2} Bl_{C_3} X)_b \cong (Bl_{C_3} Bl_{C_2} X)_b$, since both blow-ups keep track of how the two curves "collide". On the other hand, for a general choice of the three sections, the final blow-ups correspond to the blow up of different points $p \neq q$ in $(Bl_{C_2} Bl_{C_3} X)_b \cong (Bl_{C_3} Bl_{C_2} X)_b$. Now, in case the fibers are surfaces of general type (i.e. in the case of interest for us), $(Bl_{C_2}Bl_{C_3}X)_b \cong (Bl_{C_3}Bl_{C_2}X)_b$ has a finite automorhpism group. Therefore, for a general choice of the sections, the points p, q are not conjugate under any automorphism. Therefore, X^1_b is not isomorphic to X^2_b.

Since the two families agree on $B \setminus \{b\}$, and they have different central fibers, this shows that $Moduli_{\mathbf{V}}$ is not separated. As we will see later, the way to overcome this issue is both natural and beautiful.

2.2. **Properness.** In algebraic geometry, it is very common to consider a family of smooth varieites that degenerates into some singular one. This should warn us from expecting a moduli space (or functor, as we will see) to be separated on the nose.

It is well known that, for smooth curves of genus g, there is a separated coarse moduli space $\mathcal{M}_g$. On the other hand, $\mathcal{M}_g$ is not proper, as smooth curves can be deformed into singular ones. For some time, it was a problem how to possibly compactify such a space into a proper one. A choice was needed, since in general there is no unique way to complete a flat family over an affine curve to a family over its projective closure.

The problem was solved by Deligne and Mumford, that introduced the concept of stable curve, i.e. a connected curve C satisfying the following properties:

- the only singularities of C are ordinary nodes;
- the canonical sheaf ω_C is ample.

It is worth pointing out that these conditions imply that $\mathrm{Aut}(C)$ is finite. Thus, in case $g \geq 2$, there is a proper (actually projective) closure of $\mathcal{M}_g$, the so called Deligne-Mumford compactification $\overline{\mathcal{M}}_g$. Such space is a coarse moduli space for curves of genus g [Kol17][pp. 8-9]. Now, we will investigate how properness of a coarse moduli space should translate to an appropriate properness for moduli functors.

Example 2.4. The following example is about a family of elliptic curves. Smooth elliptic curves are semistable curve (a weakened version of stable curves). Still, the following example is explanatory for what happens in the stable case as well, and it has the advantage of having pretty explicit equations. A more detailed discussion of it can be found in [CM13].

Let k be an algebraically closed filed of characteristic different from 2 or 3. Then, consider the algebraic set

$$\{x^2 + y^3 + t = 0\} \subset \mathbb{A}^2 \times \mathbb{A}^1,$$

and let $f : X \to \mathbb{A}^1$ be the family induced by the closure in $\mathbb{P}^2 \times \mathbb{A}^1$. This is a flat family whose fibers are smooth elliptic curves for $t \neq 0$, and a cuspidal rational curve for $t = 0$. Hence, this is not a family of semistable curves. Indeed, the central fibers has singularities that are not simple nodes.

Thus, we would like to find a different extension of the family $g : X' \to \mathbb{A}^1 \setminus \{0\}$, obtained forgetting the central fiber. One can see that the j invariant of each fiber is 0. Thus, our expectation is that the only way to fill the family would be with another smooth elliptic curve with j invariant 0.

Although all the fibers of g are abstractly isomorphic, one can check that the family given by g is not isotrivial [CM13][pp. 20-21]. Thus, we have an obstruction to complete our family. A detailed analysis shows that the monodromy around $\{0\}$ has order 6. Thus, we hope that a branched cover of order 6 will kill the monodromy and the obstruction to extending the family.

We consider the base change

$$\begin{CD} X'' @>>> X' \\ @VVV @VVV \\ \operatorname{Spec} k[u,u^{-1}] @>>> \operatorname{Spec} k[t,t^{-1}] \end{CD}$$

induced by $t \mapsto u^6$. Thus, in affine coordinates, the new family over the puncured base is given by

$$\{x^2 + y^3 + u^6 = 0\}.$$

Since $u \neq 0$, we can perform the change of coordinates $z = xu^3$, $w = yu^2$. In these coordinates, the family is given by

$$\{z^2 + w^3 + 1 = 0\}.$$

Thus, we have an isotrivial family over $\operatorname{Spec} k[u,u^{-1}]$, and we can now extend it over the origin.

What showed in the example is consistent with the expectations of semistable reduction for curves. In particular, it shows that when there is no fine moduli space, the best we can do is to look for a valuative criterion of properness "up to finite base change". Thus, we are ready for the following definition.

Definition 2.5. Let B be a smooth curve and $B^0 \subset B$ an ope subset. Let $\pi^0 : V^0 \to B^0$ be a flat proper family of varieties in $\mathbf{V}$, i.e. $X^0 \in Moduli_{\mathbf{V}}(B^0)$. We say that $Moduli_{\mathbf{V}}$ is proper if for any such piece of data there exists a finite surjection $p : A \to B$ such taht there is an extension

$$\begin{CD} V^0 \times_B A @>>> W \\ @VVV @VV{\pi_A}V \\ B^0 \times_B A @>>> A \end{CD}$$

and $W \in Moduli_{\mathbf{V}}(A)$.

Remark 2.6. This definition corresponds to stable reduction for curves [Kol17][p. 10].

Remark 2.7. In case $Moduli_{\mathbf{V}}$ is representable, i.e. there is a fine moduli space $\mathrm{Modili}_{\mathbf{V}}$, we can take $p = \mathrm{id}_B$. In case we have a coarse moduli space, this criterion builds a correspondence "up to finite base change" between maps to the coarse moduli space and flat families.

Remark 2.8. We are aiming for functors that are both separated and proper. Thus, we are in the case when the extension after base change is unique. In particular, in the separated and proper case, the way to complete the family is essentially unique: the missing fibers are uniquely determined, and any two families $W_1 \xrightarrow{\pi_{A_1}} A_1$ and $W_2 \xrightarrow{\pi_{A_2}} A_2$ as in definition 2.5 are dominated by a third one.

2.3. **Other properties.** There are other desirable properties for a moduli functor. Some of them, such as *local closedness* [Kol17][p. 12], [HK10][pp. 124-125], are important technical assumptions. Other ones, with the same perspective of separatedness and properness, come from desired geometric properties of parameter spaces. It is worth mentioning one more of those.

Definition 2.9. A moduli functor $Moduli_{\mathbf{V}}$ is bounded if there exists a flat morphism of schemes of finte type $\pi : U \to T$ such that, for every algebraically closed field $K \supset k$, every K-scheme in $\mathbf{V}$ occurs as fiber of $U_K \to T_K$.

Remark 2.10. The above definition 2.9 is taken from [Kol17][p. 13], and it captures the more geometric meaning of boundedness. For certain purposes, a more technical definition is needed [HK10][p. 117].

3. The case of varieties of general type

Now that we have developed an appropriate language for moduli functors, we would like to discuss the correct approach to a moduli theory for varieties of general type.

3.1. **The right choice of representatives.** As showed in Example 2.3, the functor of smooth surfaces of general type is not separated. In such an example, the central fibers of the two families are birational, but not isomorphic. This is a problem that first appears in dimension 2. Therefore, we need to find a new strategy, that was completely unneeded in the case of curves.

First, we would like to discuss the case of surfaces, which already hints the correct strategy, while avoiding certain technical complications. We will exploit the Enriques-Kodaira classification of smooth surfaces [KM98][pp. 26-27]. Indeed, Castelnuovo's contractibility criterion tells us that for every birational class of smooth surfaces there is a preferable representative obtained by blowing down all (-1)-curves. This choice would indeed prevent the non-separatedness issue in Example 2.3.

The following fact shows that we are on the right track.

Proposition 3.1. *Let $f_i : X^i \to B$ be two smooth families of projective varieites over a smooth curve B. Assume that the generic fibers $X^1_{k(B)}$ and $X^1_{k(B)}$ are birational and the pluricanonical system $|mK_{X^1_{k(B)}}|$ is nonempty for some $m > 0$. Then, for every $b \in B$, the fibers X^1_b and X^2_b are birational.*

Proof. See [Kol17][p. 17, Proposition 26]. □

Thus, Proposition 3.1 hints that working up to birational class is the right approach. Also, if in addition we know that all the fibers are minimal surfaces of general type, then we would conclude that all the fibers are isomorphic. On the other hand, surfaces of general type may degenerate into surfaces of lower Kodaira dimension.

Example 3.2. We refer to [Kol17][p. 28] for a detailed discussion of this example. One can construct two smooth families of smooth projective surfaces $f_i : X^i i \to \mathbb{A}^1$, $i = 1, 2$, such that:

- X^i_t is of general type for $t \neq 0$, while X^i_0 has just nef canonical class;
- X^1 and X^2 are isomorphic over $\mathbb{A}^1 \setminus \{0\}$;
- X^1_0 and X^2_0 are isomorphic;

- X^1 and X^2 are not isomorphic.

Thus, we see that as smooth minimal surfaces of general type degenerate to a smooth minimal surface of smaller Kodaira dimension, we see that separatedness fails for the functor of smooth minimal surfaces.

As showed in Example 3.2, there is some work to do in order to understand what are the correct classes of varieties we can degenerate into in order to have a well behaved moduli functor. Furthermore, as we raise the dimension to 3 or more, we realize that we do not have a preferable smooth model within a birational class. The following proposition will give some more insight about what to do.

Proposition 3.3. *Let $f_i : X^i \to B$ be two smooth families over a smooth curve B. Assume that the canonical classes K_{X^i} are f_i-ample. Assume that the generic fibers $X^1_{k(B)}$ and $X^1_{k(B)}$ are isomorphic. Then, such an isomorphism extends to an isomorphism between X^1 and X^2.*

Proof. See [Kol17][p. 19, Proposition 29]. □

The proof of Proposition 3.3 relies on showing that the two varieties have the same relative canonical ring over the base, and that they both coincide with the relative Proj of such sheaf of algebras. This suggests a direction that will take care both of separatedness and of the generalization to higher dimension.

Indeed, thanks to the work of Birkar, Cascini, Hacon, and McKernan [HK10], we know that for a smooth variety of general type X the canonical ring $R(X, K_X) = \sum_{m\geq 0} H^0(X, \mathscr{O}_X(mK_X))$ is finitely generated. In particular, this guarantees the existence of a distinguished element $X_{can} = \text{Proj}(R(X, K_X))$, the canonical model of X, in the birational class of X. Indeed, any other smooth variety X' birational to X is such that $R(X', K_{X'}) = R(X, K_X)$. The choice of the canonical model as representative of a birational class makes so that we have a good choice in higher dimension. Also, in the spirit of Proposition 3.3, it will guarantee that our functor is separated.

Thus, following [HK10][p. 106], we can finally make a choice of functor to work with.

Definition 3.4. Let k be an algebraically closed field of characteristic 0, and $\mathfrak{Sch}_k$ the category of k-schemes. Fix $p \in \mathbb{Q}[t]$, and let $\mathcal{M}_p^{\text{smooth}} : \mathfrak{Sch}_k \to \mathfrak{Sets}$ be the following functor:

- for a k-scheme B, we have
$$\mathcal{M}_p^{\text{smooth}}(B) = \{f : X \to B | f \text{ is a smooth projective family, such that } \forall b \in B\ \omega_{X_b} \text{ is ample, and } \chi(X_b, \omega_{X_b}^{\otimes m}) = p(m)\}/\simeq,$$
where $\simeq$ denotes the relation of isomorphism over the base B;
- for a morphism $\alpha \in \text{Hom}_{\mathfrak{Sch}_k}(A, B)$, we have
$$\mathcal{M}_p^{\text{smooth}}(\alpha) = (-) \times_B \alpha.$$

We notice that, as in the case of the Hilbert scheme, we fixed a Hilbert polynomial p in Definition 3.4. The following shows that this is the right definition for our purposes.

Theorem 3.5. *The moduli functor $\mathcal{M}_p^{smooth}$ defined in Definition 3.4 is bounded, locally closed and separated. Furthermore, there is a quasi-projective coarse moduli scheme for $\mathcal{M}_p^{smooth}$.*

Proof. See [HK10][Chapter 13]. □

3.2. **How to achieve properness.** Theorem 3.5 partly solves our problem. Still, the harder part of the work is still to come. First, we have that $\mathcal{M}_p^{\text{smooth}}$ is not proper. Thus, we would like to find a family of varieties that plays the role of the stable curves introduced by Deligne and Mumford. Second, there is an important subtlety to be pointed out.

In the previous subsection, we established that the right representative in the birational class of a smooth variety of general type X is its canonical model X_{can}. Unfortunately, we omitted to recall that, in general, such canonical model X_{can} is not smooth. On the other hand, the functor of our choice $\mathcal{M}_p^{\text{smooth}}$ is so that

$$\mathcal{M}_p^{\text{smooth}}(k) = \{X | X \text{ is a smooth projective variety, such that} \\ \omega_X \text{ is ample, and } \chi(X, \omega_X^{\otimes m}) = p(m)\}.$$

This makes so that all the smooth varieties such that $X \neq X_{can}$ are not taken into account by $\mathcal{M}_p^{\text{smooth}}$. Therefore, we need to extend our functor also in order to include all X_{can} for all X smooth of general type.

As the name might suggest, canonical models of smooth varieties of general type have at worst *canonical singularities* [HK10][p. 34]. On the other hand, to make our moduli functor proper, we need to admit a class of singularities that is a generalization of the so called *log canonical* ones.

Definition 3.6. A scheme X has semi-log canonical singularities if

- ∘ X is reduced;
- ∘ X is S_2;
- ∘ K_X is $\mathbb{Q}$-Cartier;
- ∘ there exists a good semi-resolution of singularities [HK10][p. 35] $f : X' \to X$ with exceptional divisor $E = \cup E_i$, and we write $K_{X'} \equiv f^*K_X + \sum a_i E_i$ with $a_i \in \mathbb{Q}$, then $a_i \geq -1$ for all i.

Definition 3.6 is pretty technical, and figuring out it was the right one had been the main problem in this field for many years. Intuitively, semi-log canonical singularities can be thought as appropriate gluing of log canonical ones, as stable curves are appropriate gluing of smooth curves.

In this setting, the functor $\mathcal{M}_p^{\text{smooth}}$ has an extension that takes care of our concerns. Since we are dealing with families of singular varieties, there are some subtleties to take into account. These have to do with the fact that ω_{X_b} is in general just a $\mathbb{Q}$-line bundle, and therefore its higher tensor power should be reflexified to get the right object to put in the Euler characteristic. Thus, we will omit the details of the following, and refer to a more detailed exposition.

Fact 3.7. There is an extension of the moduli functor $\mathcal{M}_p^{\text{smooth}}$. Such functor is bounded, separated, proper, and locally closed. Furthermore, there is a projective coarse moduli scheme associated to it.

Proof. See [HK10][Chapter 14]. □

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