An Introduction to Abelian Varieties

Stefano Filipazzi

August 31, 2016

These notes are supposed to be a handout for the student seminar in algebraic geometry at the University of Utah. In this seminar, we will give a first introduction to abelian varieties.

1 Introduction

There are many ways and many perspectives to introduce abelian varieties. One of them is the analytic approach: how do complex, connected and compact Lie grous look like? The answer turns out to be very satisfactory: they coincide exactly with complex tori. Once this is established, the natural question an algebraic geometer would ask is about the algebraicity of complex tori. As showed by Riemann, "most" tori can not be embedded in a projective space, and there is a very beautiful and explicit description of when we succeed in finding an embedding.

A second and more algebraic approach corresponds to the following question. How do projective algebraic groups look like? In the case the ground field is $\mathbb{C}$, the answer coincides with the one found by Riemann about the algebraicity of complex tori. This motivates the definition of abelian variety over any ground field $\mathbb{K}$.

Abelian varieties are a very interesting class of varietes, subject of current research study. On the other hand, they provide many good examples and interesting results that can be discussed in an introduction to algebraic geometry. The goal of these notes is to highlight a few of these. We will prefer the analytic approach, since more intuitive and figurative.

The material for these notes is taken from [BL04], [Mum74], [Cor] and [Spe].

2 Complex tori

Consider a complex vector space $V = \mathbb{C}^g$, where $g \geq 1$, and let Λ be a lattice of full rank. This means that $\Lambda \cong \mathbb{Z}^{2g}$ as abstract group, and that $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda = V$. The subgroup Λ acts freely and properly discontinuously on V by translation; therefore, the quotient

$$X = V/\Lambda \tag{1}$$

is a complex manifold of complex dimension g. Any such is called *complex torus*. Let $(\lambda_1, \dots, \lambda_{2g})$ be a $\mathbb{Z}$-basis for Λ. Then, the set

$$\left\{ \sum_{i=1}^{2g} t_i \lambda_i | 0 \leq t_i \leq 1 \ \forall i \right\} \tag{2}$$

provides a fundamental domain for the action of Λ; therefore, X is compact.

Example 2.1. Consider $V = \mathbb{C}$, and let $\Lambda = (\lambda_1, \lambda_2)$ be any $\mathbb{R}$-basis for $\mathbb{C}$. Then $X = V/\Lambda$ is a torus from the topological point of view. It can be seen that, as we vary Λ, we obtain different complex structures.

Proposition 2.2. Any complex compact Lie group X is a complex torus.

Proof. First, we show that X is abelian. Consider the map $\psi(x, y) = xyx^{-1}y^{-1}$ given by the commutator of two elements. Fix $U \subset X$ an open neighborhood of $1 \in X$. Since $\psi(x, 1) = 1$, by continuity there exist open sets $x \in V_x$ and $1 \in W_x$ such that $\psi(V_x, W_x) \subset U$. By compactness of X, we need finitely many $V_{x_1}, \dots, V_{x_n}$ to cover X. Let $W = \cap_{i=1}^n W_{x_i}$. Then, we have $\psi(X, W) \subset U$. W.l.o.g. we may assume that U is an open chart around 1 diffeomorphic to an open ball. Thus, for any $y \in W$, $\psi_y(x) = \psi(x, y)$ is a vector-valued holomorphic function on the compact manifold X, hence constant. Since $\psi_y(1) = 1$ for any $y \in W$, we have that $\psi(X, W) = 1$. Since $\psi : X \times X \to X$ is holomorphic and constant on the non-empy open $X \times W$, it is constant. This shows X is abelian.

Now, let $\pi : V \to X$ be the universal cover. Since X is a Lie group, V is a Lie group as well. Furthermore, V has to be abelian too. Since V is simply connected, a classification result tells us that it has to be an affine space. Therefore, $V \cong \mathbb{C}^{\dim X}$. Now, we have $\Lambda = ker(\pi)$. Since it is a discrete subgroup of $\mathbb{C}^{\dim X}$, it is a lattice. Since X is compact, Λ is forced to have full rank, and therefore X is a complex torus. $\square$

3 Cohomology of complex tori

Proposition 2.2 tells us that complex tori are very concrete objects to work with. Now, a natural question to ask is how the invariants of such a complex manifold reflect this topological structure. It turns out that the cohomology groups of a complex torus $X = V/\Lambda$ have a very beautiful and explicit description.

Proposition 3.1. Let $X = V/\Lambda$ be a complex torus. Then $H_1(X; \mathbb{Z}) = \Lambda$.

Proof. By construction, $V \to X$ is the universal cover of X. Therefore $\Lambda = \pi_1(X)$. Since Λ is abelian, it coincides with $H_1(X; \mathbb{Z})$ as well. $\square$

Corollary 3.2. We have that $H^1(X; \mathbb{Z}) = Hom_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. In particular, $H^1(X; \mathbb{Z})$ is a free group of rank $2 \dim X$.

Proof. Since $H_0(X;\mathbb{Z}) = \mathbb{Z}$, then $\mathrm{Ext}^1_{\mathbb{Z}}(H_0(X;\mathbb{Z}),\mathbb{Z}) = 0$. Therefore the universal coefficient theorem guarantees us $H^1(X;\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(H_1(X;\mathbb{Z}),\mathbb{Z})$, and the claim follows. $\square$

Now, we are ready to state the main theorem about the cohomology of complex tori.

Theorem 3.3. Let $X = V/\Lambda$ be a complex torus. Then, the canonical map $\Lambda^n H^1(X;\mathbb{Z}) \to H^n(X;\mathbb{Z})$ induced by the cup product is an isomorphism for any $n \geq 1$. In particular, the cohomology ring $H^\bullet(X;\mathbb{Z})$ is isomorphic to the exterior algebra $\Lambda H^1(X;\mathbb{Z})$.

Proof. First, we notice that from the topological point of view $X \cong (S^1)^m$. Therefore, it is enough to show that the statement holds true for products of copies of S^1.

We proceed by induction on m, the case $m = 1$ being obvious. Now, assume $m \geq 2$. We regard $(S^1)^m$ as $(S^1)^{m-1} \times S^1$. The inductive hypothesis gives us a description of the cohomology of $(S^1)^{m-1}$, which is then torsion free. Therefore, the Künnet decomposition tells us

$$H^n((S^1)^m;\mathbb{Z}) \cong \bigoplus_{p+q=n} H^p((S^1)^{m-1};\mathbb{Z}) \otimes H^q(S^1;\mathbb{Z}). \tag{3}$$

Now, the inductive hypothesis guarantees that

$$\bigoplus_{p+q=n} H^p((S^1)^{m-1};\mathbb{Z}) \otimes H^q(S^1;\mathbb{Z}) \cong \bigoplus_{p+q=n} \Lambda^p H^1((S^1)^{m-1};\mathbb{Z}) \otimes \Lambda^q H^1(S^1;\mathbb{Z}). \tag{4}$$

On the other hand, the Künneth decomposition tells us

$$H^1((S^1)^m;\mathbb{Z}) \cong H^1((S^1)^{m-1};\mathbb{Z}) \oplus H^1(S^1;\mathbb{Z}), \tag{5}$$

which leads to

$$\Lambda^n H^1((S^1)^m;\mathbb{Z}) \cong \bigoplus_{p+q=n} \Lambda^p H^1((S^1)^{m-1};\mathbb{Z}) \otimes \Lambda^q H^1(S^1;\mathbb{Z}). \tag{6}$$

Now, we just have to put these isomorphisms together. By the inductive hypothesis, the isomorphism 4 is given by the cup product. Since the Künneth decomposition respects cup product, the induced isomorphism

$$\Lambda^n H^1((S^1)^m;\mathbb{Z}) \cong H^n((S^1)^m;\mathbb{Z}) \tag{7}$$

is given by the cup product as well. This concludes the inductive step.

The second part of the statement is just a formal consequence of what just proved. $\square$

Observation 3.4. If we consider the cohomology ring with coefficients in $\mathbb{C}$, namely $H^\bullet(X;\mathbb{C})$, we can find a more explicit description via differential forms. Introduce complex coordinates $(z_1,\ldots,z_g)$ on V. Then, the holomorphic differentials dz_i are Λ-invariant. Therefore, they descend to holomorphic $(1,0)$-forms on X. By the maximum modulus principle, any holomorphic function on X is constant. Therefore the dz_i's are closed but not exact forms. Analogously, the $(0,1)$-forms $d\bar{z}_i$ are closed but not exact.

The Dolbeault isomorphism tells us that $(dz_1, \dots, dz_g, d\bar{z}_1, \dots, d\bar{z}_g)$ is a basis for $H^1(X;\mathbb{C})$. Now, it is well known that the cup product of cocycles corresponds to the wedge product of forms under the de Rham isomorphism. Therefore, a basis for $H^\bullet(X;\mathbb{C})$ is given by

$$dz_{i_1} \wedge \dots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}, \tag{8}$$

where $|I_p| + |I_q| \leq 2g$. In particular, this way of writing cohomology classes of X provides the Hodge decomposition of $H^\bullet(X,\mathbb{C})$.

4 Riemann bilinear relations

Now, we are ready to discuss whether complex tori are algebraic varieties or not. The question we have to address is the following. Can we embed a complex torus $X = V/\Lambda$ into a suitable projective spave $\mathbb{P}^N$? First, we should point out a topological property of projective varieties.

Proposition 4.1. Let Y be a smooth projective variety; then, Y has a complex subvariety of (complex) codimension 1 whose homology class is nontrivial.

Proof. The idea is to detect that an homology class is non-zero via the intersection product on homology classes[1]. In particular, we will use the following useful fact. If A and B are two subvarieties of complementary dimension, and they meet properly (i.e. transversally), then $A \cap B$ is a finite set of points. Furthermore, under the same assumptions, $[A] \cdot [B] = \sharp(A \cap B)$.

Now, assume Y has dimension n and is embedded in $\mathbb{P}^N$. Our goal is to find a subvariety $A \subset Y$ of dimension $n-1$ whose homology class is non-zero. Therefore, we will try to find a curve $B \subset Y$ such that $[A] \cdot [B] \neq 0$. Pick a point $P \in Y$, and consider $H \subset \mathbb{P}^N$ a hyperplane through P. If H is generic (e.g. it is not tangent to Y), then $A = Y \cap H$ is a smooth hypersurface (i.e. a subvariety of codimension 1) in Y. Analogously, if we consider a generic $(N-n+1)$-plane L through P, $B = L \cap Y$ is a smooth curve in Y. For a suitable choice of H and L, A and B meet properly. In particular, we have $[A] \cdot [B] \neq 0$. This shows that the cohomology $[A] \in H_{2n-2}(Y;\mathbb{Z})$ is non-zero. $\square$

Now, a complex torus $X = V/\Lambda$ is a smooth manifold. Our goal is to show that, for a general choice of a lattice Λ, X is not algebraic. The strategy is to use Proposition 4.1. For simplicity, we will consider the case $V = \mathbb{C}^2$

Now, let (z, w) be complex coordinates on $\mathbb{C}^2$. As above mentioned, dz and dw are holomorphic 1-forms on X. As remarked in Observation 3.4, $\omega = dz \wedge dw$ is a non-zero holomorphic 2-form. Now, consider a complex curve $Y \subset X$. Thus,

[1] Since Y is smooth, we can apply Poincaré duality $P : H_i(Y;\mathbb{Z}) \to H^{2\dim Y - i}(Y;\mathbb{Z})$. Then, for two subvarieties A and B of complementary dimension, we set $[A] \cdot [B] = [A] \cap P([B])$.

$[Y] \in H_2(X;\mathbb{Z})$. Since Y has complex dimension 1, there are no holomorphic 2-forms on Y[2]. This tells us that $\omega_{|Y} = 0$. In particular, we have

$$[Y] \cap \omega = \int_Y \omega = 0. \tag{9}$$

Now, let (a,b,c,d) be a basis for Λ. We can visualize it as a matrix

$$P = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}. \tag{10}$$

As explained in Theorem 2.2, $(a \wedge b, a \wedge c, a \wedge d, b \wedge c, b \wedge d, c \wedge d)$ is a basis for $H^2(X;\mathbb{Z})$. If we pair ω with such a basis, we extract the six maximal minors of P (this is just the formula for the area of the parallelogram!).

If Λ is chosen generically, no $\mathbb{Z}$-linear combination of the above minors is zero, except the trivial one. In particular, this means that any non-zero element $\alpha \in H_2(X;\mathbb{Z})$ is such that $\alpha \cap \omega \neq 0$. By Proposition 4.1, we can conclude that, for a generic choice of Λ, the complex torus $X = \mathbb{C}^2/\Lambda$ is not projective.

Now, we are ready to provide the theorem that answers the question about projectivity of complex tori.

Theorem 4.2 (Riemann Bilinear Relations)**.** Let V be a complex vector space of dimension g, and Λ a lattice of full rank. Fix a basis $(e_1,\dots,e_g)$ for V, and a basis $(\lambda_1,\dots,\lambda_{2g})$ for Λ. Let P be the period matrix of Λ, i.e. the $g \times 2g$ matrix such that $X \cong \mathbb{C}^g/P\mathbb{Z}^{2g}$. Then, X is projective if and only if there is a non-degenerate alternating matrix $E \in M_{2g}(\mathbb{Z})$ such that

i $PE^{-1}P^T = 0$;

ii $iPE^{-1}\bar{P}^T > 0$.

The two relations in Theorem 4.2 are called first and second Riemann bilinear relations. A complex torus satisfying them is called *Abelian variety.* As we saw with the previous example, most complex tori are not abelian varieties.

It is interesting to consider one extreme case, namely the case of complex dimension 1. We will see that, in this case, all complex tori are projective: they are called *elliptic curves.* Now, we have $V = \mathbb{C}$, and Λ is generated by two complex numbers a and b. Then, we have

$$P = \begin{pmatrix} a & b \end{pmatrix}. \tag{11}$$

Since a and b are linearly independent over $\mathbb{R}$, we have $\mathrm{Im}(a\bar{b}) \neq 0$. Then, for E we have two candidates, namely

$$E_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{12}$$

[2]Along Y, either one among z and w will be the independent complex variable. For instance, say that locally the implicit function theorem tells us $w = f(z)$. Then, along Y, $dw = \frac{df}{dz}dz$. This gives that, locally along Y, $\omega_{|Y} = \frac{df}{dz}dz \wedge dz = 0$.

Also, we have that $E_2 = E_1^{-1}$. Now, the first Riemann relation is taken care by

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0. \tag{13}$$

Now, we have

$$i \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = \pm 2\mathrm{Im}(a\bar{b}). \tag{14}$$

Therefore, choosing the suitable E_i so that $\pm 2\mathrm{Im}(a\bar{b}) > 0$, we conclude that the second Riemann bilinear relation is satisfied as well.

5 Line bundles on complex tori

There are many features of complex tori that can be understood in terms of their universal cover. One of the most beautiful and relevant is related to the description of line bundles. Before moving on, we recall what a line bundle it.

Definition 5.1. Given a complex manifold X, a holomorphic vector bundle E of rank r on X is a complex manifold E together with a submersion $p : E \to X$ such that

- $p^{-1}(x) \cong \mathbb{C}^r$ for any $x \in X$;
- for any $x \in X$ there is an open neighborhood $U \subset X$ of x and an isomorphism of complex manifolds η such that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\eta"] \arrow[dr, "p"] & & U \times \mathbb{C}^r \arrow[dl] \\ & U & \end{tikzcd}$$

  commutes, and $\eta_{|p^{-1}(y)}$ is an isomorphism of $\mathbb{C}$-vector spaces for any $y \in U$.

In case $r = 1$, we say E is a line bundle.

Observation 5.2. This definition works in any category: if we are interested in topological (smooth) manifolds, replace holomorphic with continuous (smooth). In the case we are interested in algebraic varieties, holomorphic will be replaced by algebraic (i.e. qutients of polynomial functions). Furthermore, in the algebraic setting, we have to consider the Zariski topology of X instead of the Euclidean one.

Now, we can try to understand line bundles on complex tori. Let $X = V/\Lambda$ be a complex torus, and $p : L \to X$ a line bundle. Then, a natural operation on vector bundles is the pullback along a morphism. In this case, we have that $\pi^* L$ sits in a Cartesian square

$$
\begin{tikzcd}
\pi^* L \arrow[r, "\varphi"] \arrow[d, "q"] & L \arrow[d, "p"] \\
V \arrow[r, "\pi"] & X
\end{tikzcd}
$$

Since V is a contractible space, any line bundle on it is topologically trivial, i.e. homeomorphic to $V \times \mathbb{C}$. Is it still true if we work in the analytic category, i.e. if we allow just holomorphic maps?

On a line bundle, we know that the transition functions from one trivialization to the other are non-vanishing holomorphic functions satisfying the cocycle condition. Therefore, Čech cohomology tells us that $H^1(Y, \mathcal{O}_Y^*)$ parametrizes holomorphic line bundles up to isomorphism on a complex manifold Y[3]. The second Cousin's problem asks about the non-vanishing of the group $\mathcal{O}_Y^*$ for a complex manifold Y. Here we will use the following.

Fact 5.3. $H^1(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = 0$.

We will use Fact 5.3 to show the following.

Proposition 5.4. Any holomorphic line bundle on $\mathbb{C}^n$ is trivial.

Proof. We consider the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{\mathbb{C}^n} \xrightarrow{\exp(2\pi\sqrt{-1}\cdot)} \mathcal{O}_{\mathbb{C}^n}^* \to 1. \tag{15}$$

This exact sequence of sheaves provides a long exact sequence in cohomology. In particular, we get

$$\ldots \to H^1(\mathbb{C}^n, \mathbb{Z}) \to H^1(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \to H^1(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}^*) \to H^2(\mathbb{C}^n, \mathbb{Z}) \to \ldots \tag{16}$$

The sheaf cohomology in the analytic topology of the locally constant sheaf $\mathbb{Z}$ agrees with the usual topological cohomology with coefficients in $\mathbb{Z}$. Therefore we have $H^2(\mathbb{C}^n, \mathbb{Z}) = H^1(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = 0$. Therefore, by exactness, we have $H^1(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}^*)$, and the claim follows. $\square$

Now, we know that the above commutative diagram can be written, up to isomorphism, as

$$
\begin{tikzcd}
\mathbb{C}^g \times \mathbb{C} \arrow[r, "\varphi"] \arrow[d, "q"] & L \arrow[d, "p"] \\
\mathbb{C}^g \arrow[r, "\pi"] & X
\end{tikzcd}
$$

[3]This is true more generally: here $\mathcal{O}_Y^*$ denotes the sheaf of non-vanishing analytic functions on Y. If Z is an algebraic variety, and we consider $\mathcal{O}_Z^*$ as the sheaf of non-vanishing algebraic functions on Z, then $H^1(Z, \mathcal{O}_Z^*)$ parametrizes algebraic line bundles on Z.

Therefore, we can regard L as a quotient of $\mathbb{C}^g \times \mathbb{C}$. In particular, we have

$$(z, \xi) = (z + u, f_u(z)\xi), \tag{17}$$

where $z \in \mathbb{C}^g$, $\xi \in \mathbb{C}$ and $u \in \Lambda$. Furthermore, the functions f_u are nowhere vanishing holomorphic functions satisfying the cocylce condition

$$f_u(z + v) \cdot f_v(z) = f_{u+v}(z). \tag{18}$$

It can be showed that, up to choosing a different trivialization for π^*L, we can write the functions f_u as

$$f_u(z) = \exp(2\pi\sqrt{-1}(a_u(z) + b_u)), \tag{19}$$

where a_u is a linear form, and b_u is a constant. Then, the cocylce condition can be guaranteed by setting

$$a_{u+v}(z) = a_u(z + v) + a_v(z) \tag{20}$$

$$b_{u+v} \equiv b_u + b_v + a_u(v) \mod \mathbb{Z}. \tag{21}$$

Condition (20) makes so that we can extend $a_u(z)$ to a bilinear form $A(z, w)$ that is $\mathbb{C}$-linear in the first entry and $\mathbb{R}$-linear in the second (remember that $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda = V$). Condition (21) is symmetric in u and v, and can be now interpreted as

$$E(u, v) := A(u, v) - A(v, u) \in \mathbb{Z}. \tag{22}$$

Then, we set

$$c_u = b_u - \frac{1}{2}A(u, u). \tag{23}$$

This makes so that

$$c_u + c_v \equiv c_{u+v} + \frac{1}{2}E(u, v) \mod \mathbb{Z}. \tag{24}$$

Then, we define $\rho(u) = \exp(2\pi\sqrt{-1}c_u)$; this, together with identities (22) and (24), tells us

$$\rho(u) \cdot \rho(v) = \pm\rho(u + v). \tag{25}$$

Furthermore, the transition functions $f_u(z)$ can now be expressed as

$$f_u(z) = \rho(u) \exp\left(2\pi\sqrt{-1}\left(A(z, u) + \frac{A(u, u)}{2}\right)\right). \tag{26}$$

With some more work, we can reduce to the case when $|\rho(u)| = 1$ (i.e. ρ is a quasi-character on Λ) and the cocylce functions are

$$f_u(z) = \rho(u) \exp\left(\pi H(z, u) + \frac{\pi}{2}H(u, u)\right), \tag{27}$$

where H is an Hermitian form.

The astonishing result is that the Hermitian form H and the quasi-character ρ carry all the information about the line bundle L.

Theorem 5.5 (Appell-Humbert). *Let $X = V/\Lambda$ be a complex torus of dimension g. Then*

- any line bundle L on X can be expressed as $L(H, \rho)$, where H is an Hermitian form on V and ρ is a quasi-character on Λ;
- $L(H_1.\rho_1)$ and $L(H_2, \rho_2)$ are isomorphic if and only if $H_1 = H_2$ and $\rho_1 = \rho_2$;
- $E = \mathrm{Im}(H)$ is the Chern class of $L(H, \rho)$.

References

[BL04] C. Birkenhake and H. Lange. *Complex Abelian Varieties*. Ed. by Springer-Verlag. 2nd ed. Vol. 302. Grundlehren der mathematischen Wissenschaften. Berlin, Heidelberg, 2004.

[Cor] M.D.T. Cornalba. *Lezioni sulle superficie di Riemann e i tori complessi*. URL: `http://mate.unipv.it/cornalba/dispense/LezSupRiemann.pdf` (visited on 08/30/2016).

[Mum74] D. Mumford. *Abelian Varieties*. Ed. by Oxford University Press. 2nd ed. Vol. 5. Tata Institute of Fundamental Research Studies in Mathematics. Bombay, 1974.

[Spe] D. Speyer. *Secret Blogging Seminar*. URL: `https://sbseminar.wordpress.com/2008/02/14/complex-manifolds-which-are-not-algebraic/` (visited on 08/30/2016).