Flag Varieties

Matthew Goroff

November 2, 2016

1. Grassmannian Variety

Definition 1.1: Let V be a k-vector space of dimension n. The **Grassmannian** $Gr(r, V)$ is the set of r-dimensional subspaces of V.

It is clear that $Gr(1, V)$ is equivalent to $\mathbb{P}(V)$, and thus is a projective variety. We will show that all Grassmannians are in fact projective varieties. To do this we will use the Plücker embedding:

$$\phi \colon Gr(r, V) \to \mathbb{P}\left(\bigwedge^r(V)\right)$$

$$U \mapsto k(u_1 \wedge u_2 \wedge \cdots \wedge u_r)$$

where $\{u_1, \ldots, u_r\}$ is a basis for U.

Proposition 1.2: $Gr(r, V)$ is a projective variety.
Proof: First we show the Plücker embedding ϕ is well-defined.
If $\{u_1 \wedge \cdots \wedge u_r\}, \{u'_1 \wedge \cdots \wedge u'_r\}$ are two basis for U, then there exists a change of basis matrix M relating these two, and we have $u_1 \wedge \cdots \wedge u_r = \det(M)(u'_1 \wedge \cdots \wedge u'_r)$. These are equal in projective space, so ϕ is well-defined.
Now we show that the Plücker embedding is in fact an embedding.
Let $W \in Gr(r, V)$ with basis $\{w_1, \ldots, w_r\}$, and set $w = w_1 \wedge \cdots \wedge w_r$. Define the map $\varphi_w \colon V \to \bigwedge^{r+1}(V)$ by $\varphi_w(v) = v \wedge w$. It is clear that φ_w is a linear map, and that $W \subseteq \ker(\varphi_w)$. To show the reverse inclusion we extend to the basis $\{w_1, \ldots, w_n\}$ of V and consider $v = \sum_{i=1}^n \alpha_i w_i$.

$$\begin{aligned} v \wedge w &= \sum_{i=1}^{n} \alpha_i w_i \wedge w \\ &= \sum_{i=r+1}^{n} \alpha_i w_i \wedge w \\ &= \sum_{i=r+1}^{n} \alpha_i w_i \wedge w_1 \wedge \cdots \wedge w_r \\ &= \sum_{i=r+1}^{n} (-1)^r \alpha_i w_1 \wedge \cdots \wedge w_r \wedge w_i \end{aligned}$$

As each $w_1 \wedge \cdots \wedge w_r \wedge w_i$ is unique for each $i > r$, if $v \wedge w = 0$ then $\alpha_i = 0$ for all $i > r$. This implies that if $v \in \ker(\varphi_w)$ then $v = \sum_{i=1}^r \alpha_i w_i$, so $v \in W$. So we have that $W = \ker(\varphi_w)$.
Now let $U \in Gr(r, V)$ with basis $\{u_1, \ldots, u_r\}$, set $u = u_1 \wedge \cdots \wedge u_r$. If $\phi(W) = \phi(U)$ then $kw = ku$, thus $\ker(\varphi_w) = \ker(\varphi_u)$. So from above we have that $W = U$, making ϕ an embedding.

Aside: Notice that if we fix a basis $\{v_1, \dots, v_n\}$ for V, then we get a basis for $\bigwedge^r(V)$. So given a $W \in Gr(r, V)$, we can use the Plücker embedding to find the coordinates of W in $\mathbb{P}^N$.

Example 1.3: Let V have basis $\{v_1, v_2, v_3\}$, and let $W \in Gr(2, V)$. Then W has some basis $\{w_1, w_2\}$ which we can write in terms of the basis of V: $w_1 = a_1v_1 + a_2v_2 + a_3v_3$ and $w_2 = b_1v_1 + b_2v_2 + b_3v_3$. So:

$$\begin{aligned} w_1 \wedge w_2 &= (a_1v_1 + a_2v_2 + a_3v_3) \wedge (b_1v_1 + b_2v_2 + b_3v_3) \\ &= (a_1b_2 - a_2b_1)v_1 \wedge v_2 + (a_1b_3 - a_3b_1)v_1 \wedge v_3 + (a_2b_3 - a_3b_2)v_2 \wedge v_3 \end{aligned}$$

So the Plücker coordinates are: $(a_1b_2 - a_2b_1, a_1b_3 - a_3b_1, a_2b_3 - a_3b_2)$.
If we let $w = w_1 \wedge w_2$ then we can express the map φ_w in matrix form:

$$\begin{pmatrix} a_2b_3 - a_3b_2 & -(a_1b_3 - a_3b_1) & a_1b_2 - a_2b_1 \end{pmatrix}$$

Example 1.4: Let V have basis $\{v_1, v_2, v_3\}$, and let $W \in Gr(1, V)$. Then W has some basis $\{w\}$ which we can write as $w = a_1v_1 + a_2v_2 + a_3v_3$. With Plücker coordinates: (a_1, a_2, a_3). We can express the map φ_w in matrix form:

$$\begin{pmatrix} a_2 & -a_1 & 0 \\ a_3 & 0 & -a_1 \\ 0 & a_3 & -a_2 \end{pmatrix}$$

Now we prove that the image of the Plücker embedding is Zariski closed.
Consider $v \in V$ and $w \in \bigwedge^r(V)$. We say that v divides w if $w = v \wedge u$ for some $u \in \bigwedge^{r-1}(V)$.
Notice that v divides w implies that $v \wedge w = 0$ as then w must be the wedge product of some scale of v. We will show that $v \wedge w = 0$ implies that v divides w. Choose a basis for V that includes v, and write w in the induced basis on $\bigwedge^r(V)$. Now when we take $v \wedge w$, we see that each term either has two v's or is a basis element of $\bigwedge^{r+1}(V)$. As the basis elements cannot cancel with each other, if $v \wedge w = 0$ then the only terms of w that occur are those that contain a v. Thus v divides w.
Now consider $\ker(\varphi_w)$. We have just shown that if $v \in \ker(\varphi_w)$ then there exists some $u \in \bigwedge^{r-1}(V)$ such that $w = v \wedge u$. So let $w_1, \dots, w_t \in \ker(\varphi_w)$ be linearly independent vectors in V. Then we have that $w = w_1 \wedge \cdots \wedge w_t \wedge u$ for some $u \in \bigwedge^{r-t}(V)$. Thus $t \leqslant r$, so $\dim(\ker(\varphi_w)) \leqslant r$. So $\mathrm{rank}(\varphi_w) \geqslant n - r$.
But also notice that if $w = w_1 \wedge \cdots \wedge w_r$ (i.e. w is a decomposable element) then by above we have that $\mathrm{rank}(\varphi_w) = n - r$.
This gives us that w is decomposable if and only if $\mathrm{rank}(\varphi_w) \leqslant n - r$.
Recall that the image we are concerned with are exactly the scalers of decomposable elements. Thus we have shown that an element kw is in the image if and only if $\mathrm{rank}(\varphi_w) \leqslant n - r$. This is a polynomial condition on the matrix of φ_w, and we know that the matrix is formed of (up to sign) the Plücker coordinates of w. Thus whether or not an element is in the image is a polynomial condition on it's coordinates, meaning that the image is Zariski closed. □

2. Flag Varieties

Definition 2.1: Let V be a finite dimensional vector space. A **flag** is a nested sequence of subspaces of V:

$$V_1 \subset V_2 \subset \cdots \subset V_r$$

The **signature** of a flag is the set of dimensions of the subspaces: $(\dim(V_1), \ldots, \dim(V_r))$.

Definition 2.2: Let V be a finite dimensional vector space. A **flag variety** is the set of all flags of a particular signature. We write:

$$\mathcal{F}(V; n_1, \ldots, n_r) = \{V_1 \subset \cdots \subset V_r \mid \dim(V_i) = n_i\}$$

A flag variety of the form $\mathcal{F}(V; 1, \ldots, n)$ is called a **complete flag variety**. Otherwise we say that $\mathcal{F}(V; n_1, \ldots, n_r)$ is a **partial flag variety**.

Proposition 2.3: Let V be a vector space and $0 < n_1 < \cdots < n_r \leqslant n$, then $\mathcal{F}(V; n_1, \ldots, n_r)$ is a projective variety.

Proof: There is an obvious embedding:

$$\Psi\colon \mathcal{F}(V; n_1, \ldots, n_r) \to Gr(n_1, V) \times \cdots \times Gr(n_r, V)$$

We will show that the image is Zariski closed.

Let $\pi_{ij}\colon Gr(n_1, V) \times \cdots \times Gr(n_r, V) \to Gr(n_i, V) \times Gr(n_j, V)$ be the projection for $i < j$.

Notice that:

$$\Psi(\mathcal{F}(V; n_1, \ldots, n_r)) = \bigcap_{i<j} \pi_{ij}^{-1}(\Psi(\mathcal{F}(V; n_i, n_j)))$$

So it is sufficient to show that for all $i < j$, $\Psi(\mathcal{F}(V; r, s))$ is closed.

Let $\{v_1, \ldots, v_n\}$ be a basis for V and let $(U, W) \in Gr(r, V) \times Gr(s, V)$. Now let $\{u_1, \ldots, u_r\}$ be a basis for U and $\{w_1, \ldots, w_s\}$ be a basis for W. Now set $u = u_1 \wedge \cdots \wedge u_r$ and $w = w_1 \wedge \cdots \wedge w_s$. We have the maps φ_u and φ_w from before, and we can construct the map:

$$\varphi_u \oplus \varphi_w \colon V \to \bigwedge^{r+1}(V) \oplus \bigwedge^{s+1}(V)$$

From before we know that $\ker(\varphi_u) = U$ and $\ker(\varphi_w) = W$, so it is clear that $\ker(\varphi_u \oplus \varphi_w) = U \cap W$. So we have that:

$$\begin{aligned} \operatorname{rank}(\varphi_u \oplus \varphi_w) &= \dim(V) - \dim(\ker(\varphi_u \oplus \varphi_w)) \\ &= \dim(V) - \dim(U \cap W) \\ &\geqslant \dim(V) - \dim(U) = n - r \end{aligned}$$

This implies that $U \subset W$ if and only if $\operatorname{rank}(\varphi_u \oplus \varphi_w) = n - r$ if and only if $\operatorname{rank}(\varphi_u \oplus \varphi_w) \leqslant n - r$. As before, we can represent $\varphi_u \oplus \varphi_w$ by a matrix with respect to a basis, and see that the entries are Plücker coordinates (up to sign) of u and w. $\operatorname{rank}(\varphi_u \oplus \varphi_w) \leqslant n - r$ is a polynomial condition on this matrix, and thus on the coordinate of u and v, thus we have that $U \subset W$ exactly at the zeros of a set of polynomials. So $\Psi(\mathcal{F}(V; r, s))$ is Zariski closed. $\square$

Example 2.4: Let V be a 3-dimensional vector space with basis $\{v_1, v_2, v_3\}$. Let $(U \subset W) \in \mathcal{F}(V; 1, 2)$ with $\{u\}$ a basis for U and $\{w_1, w_2\}$ a basis for W. We see that:

$$U \subset W \iff \ker(\varphi_u \oplus \varphi_w) = U$$

So if we let $t = t_1v_1 + t_2v_2 + t_3v_3$ then we have that:

$$U \subset W \iff t_1b_{23} - t_2b_{13} + t_3b_{12} = 0$$

A homogeneous polynomial.

3. Algebraic Groups

Definition 3.1: Let G be a group with the structure of an affine algebraic variety such that the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are regular maps of algebraic varieties. Then G is a **linear algebraic group**.

Example 3.2: $Gl(n, k)$ is an algebraic group.
It is the complement of the zero set of the determinant, a single polynomial. Thus it is an affine variety. The formulas for matrix multiplication and inverse can easily be seen as polynomial equations (as $\det \neq 0$).

Definition 3.3: If G is an algebraic group, then we call any maximal connected solvable closed subgroup a **Borel subgroup**.

Example 3.4: Consider $T(n, k)$, the set of upper-triangular invertible matrices.
$T(n, k)$ is the zero set of the polynomials $x_{ij} = 0$ for $i > j$ in $Gl(n, k)$. So it is also an algebraic group, thus it is Zariski closed. $T(n, k)$ is also connected and solvable, so if it were the maximal such subgroup in $Gl(n, k)$ then it would be a Borel subgroup.
The Lie-Kolchin theorem tells us that if H is a connected solvable subgroup of $Gl(n, k)$, then H is conjugate to a subgroup of $T(n, k)$. Thus $T(n, k)$ is maximal among connected solvable subgroups of $Gl(n, k)$ so it is a Borel subgroup.

Definition 3.5: Let G be an algebraic group and V a variety. we say that G **acts on** V if there is a group action $G \times V \to V$ that is also a regular map of algebraic varieties.

Just as we can study orbits and stabilizers of group actions, we can study them on algebraic group actions as well. Consider the group action of $Gl(n, k)$ where $A \in Gl(n, k)$

Proposition 3.6: $Gl(n, k)$ acts transitively on flag varieties.
Proof: Let $(U_{n_1} \subset \cdots \subset U_{n_r}), (W_{n_1} \subset \cdots \subset W_{n_r}) \in \mathcal{F}(V; n_1, \ldots, n_r)$. Choose basis $\{u_1, \ldots, u_n\}$ and $\{w_1, \ldots, w_n\}$ of V such that $\{u_1, \ldots, u_{n_i}\}$ is a basis for U_{n_i} and $\{w_1, \ldots, w_{n_i}\}$ is a basis for W_{n_i}. We can pick an $A \in Gl(n, k)$ such that $A(u_i) = w_i$, a change of basis matrix. $\square$

Proposition 3.7: $T(n, k)$ stabilizes complete flags.

Definition 3.8: Let G be an algebraic group. If P is a closed subgroup of G where G/P is a projective variety then we call P a **parabolic** subgroup. Notice that P contains a Borel subgroup.

Proposition 3.9: Every parabolic subgroup of $Gl(n, k)$ is the stabilizer of some flag.
Proof: Let P be a parabolic subgroup. A calculation shows that P is in block matrix form with r blocks.
Let $\mathcal{F}(V; n_1, \ldots, n_r)$ be the flag variety with subscripts the same as for P. Let $\{e_1, \ldots, e_n\}$ be a basis for V, and set F to be the flag $(V_{n_1} \subset \cdots \subset V_{n_r})$ where V_{n_i} is the span of the first n_i of the e's. It is clear that P stabilizes F. $\square$

References:

1) Letz, Janina *Flag Varieties* 2015.
2) Morandi, Patrick *Algebraic Groups, Grassmannians, and Flag Varieties* 1998.