MODULAR CURVES AND MODULAR FORMS

All of the material covered here can be found in the references, though I claim responsibility for mistakes. As these are very rough notes, I have not been diligent in adding citations for each specific results. The advanced reader should consult [1].

1. Basic Theory of Elliptic Curves

Elliptic curves admit numerous descriptions: projective curves of genus 1 with a distinguished point, varieties with equations $y^2 = x^3 + ax + b$, complete group varieties of dimension 1, etc. All are equivalent and it is instructive to see how one goes from one object to the next. Our goal is to classify elliptic curves up to isomorphism. Given a field $K$, we will denote by $\text{Ell}(K)$ to be the set of elliptic curves over $K$ (i.e. the equation defining the curve has coefficients in $K$) up to isomorphism. We consider curves $E_1, E_2$ over $K$ to be isomorphic if there is a map $\varphi: E_1 \to E_2$ defined over $K$ (meaning the coefficients of $\varphi$ are in $K$). This is an example of a moduli problem.

We begin by studying $\text{Ell}(\mathbb{C})$. Elliptic curves over $\mathbb{C}$ have a particularly nice form:

**Proposition 1.** For every elliptic curve $E/\mathbb{C}$, there is a lattice $\Lambda_E \subset \mathbb{C}$ and a complex analytic group isomorphism $\mathbb{C}/\Lambda_E \to E(\mathbb{C})$. Further, maps $E \to E'$ correspond to complex numbers $\alpha$ such that $\alpha \Lambda_E \subset \Lambda_{E'}$.

In particular two elliptic curves $E, E'$ are isomorphic if and only if $\alpha \Lambda_E = \Lambda_{E'}$ for some $\alpha \in \mathbb{C}$, i.e. if the lattices $\Lambda_E, \Lambda_{E'}$ are homothetic. Therefore $\text{Ell}(\mathbb{C})$ is the same as lattices in $\mathbb{C}$ up to homothety, which are easier to classify.

In classifying lattices up to homothety, we need only consider lattices of the form $\Lambda_\tau = \mathbb{Z} + \tau \mathbb{Z}$ for $\tau$ in the upper half plane $\mathbb{H} = \{ z \in \mathbb{C} : \text{im}(z) > 0 \}$. Showing that any lattice $\Lambda$ is homothetic to $\Lambda_\tau$ for some $\tau \in \mathbb{H}$ is very easy. Thus we have a list of lattices parameterized by $\mathbb{H}$ containing representatives for each homothety class. This list however is redundant. The possible bases of $\Lambda_\tau$ are $\{ c \tau + d, a \tau + b \}$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. Hence the lattices enumerated by $\tau, \gamma \tau = \frac{ac+bd}{c\tau+d}$ are homothetic since the lattice $\Lambda_\gamma = (a \tau + b, c \tau + d)$ and the latter is homothetic to $\Lambda_{\gamma \tau}$ by multiplying by $\frac{1}{c\tau + d}$ (in particular the isomorphism $\mathbb{C}/\Lambda_\tau \to \mathbb{C}/\Lambda_{\gamma \tau}$ is $z \mapsto \frac{z}{c\tau + d}$). This defines an action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, and we see that $\Lambda_\tau, \Lambda_{\tau'}$ are homothetic for $\tau, \tau' \in \mathbb{H}$ if and only if $\tau, \tau'$ have the same orbit under the action of $\text{SL}_2(\mathbb{Z})$. In short, we have produced a bijection between $\mathbb{H}/\text{SL}_2(\mathbb{Z})$ and lattices up to homothety, which we already know is in bijection with $\text{Ell}(\mathbb{C})$. Explicitly, the map is given by

$$\mathbb{H}/\text{SL}_2(\mathbb{Z}) \cong \text{Ell}(\mathbb{C})$$

$$\tau \mapsto \mathbb{C}/\Lambda_\tau$$

$$\omega_1 \omega_2 \leftrightarrow \mathbb{C}/\langle \omega_1, \omega_2 \rangle$$

The space $\mathbb{H}/\text{SL}_2(\mathbb{Z})$ can be described explicitly using a fundamental domain. To describe this region, we use the fact that $\text{SL}_2(\mathbb{Z})$ is generated by the matrices $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. These correspond to the maps $\tau \mapsto \frac{-1}{\tau}$ and $\tau \mapsto \tau + 1$. I'm a little bit too lazy to tex up the fundamental region, but the interested reader may look at the wikipedia article on the $j$-invariant, which has a nice figure. Alternatively, there is a nice picture in [3] in the section on modular functions. From the picture, we can see that $\mathbb{H}/\text{SL}_2(\mathbb{Z})$ is homeomorphic to $\mathbb{C}$. In fact, much more is true.
So far we have given a bijection between two sets, which is really not very interesting data. We would like to give \( H/\text{SL}_2(\mathbb{Z}) \) the structure of an open Riemann surface. The complex structure on \( H \) can be used to give \( H/\text{SL}_2(\mathbb{Z}) \) a complex structure, though one must be careful defining the complex structure at the points \( e^{\pi i/3}, e^{2\pi i/3}, i \). Given the complex structure on \( H/\text{SL}_2(\mathbb{Z}) \) we want to show that \( H/\text{SL}_2(\mathbb{Z}) \) is isomorphic to the complex points of an affine curve. To do this we use the \( j \)-invariant. We can define the \( j \)-invariant either for lattices \( \Lambda \) or for curves \( E \) of the form \( y^2 = x(x-1)(x-\lambda) \). In the latter case we have

\[
j(E) = \frac{2^8 \lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}
\]

We will not pursue this further, rather we will look at the \( j \)-invariant of a lattice \( \Lambda \). We first need to define functions \( G_2k(\Lambda) = \sum_{\omega \in \Lambda^*} \omega^{-2k} \) which converge absolutely for \( k \geq 1 \). The functions \( g_2 = 60G_4 \) and \( g_3 = 140G_6 \) are important to define the discriminant and \( j \)-invariant:

\[
\Delta(\Lambda) = g_3^3(\Lambda) - 27g_3^2(\Lambda), \quad j(\Lambda) = \frac{g_3^3(\Lambda)}{\Delta(\Lambda)}
\]

The quantities \( g_2 \) and \( g_3 \) are important as \( C/\Lambda \) is isomorphic to the elliptic curve \( y^2 = 4x^3 + g_2(\Lambda)x + g_3(\Lambda) \). The key property of the \( j \) invariant is that \( j(\Lambda) = j(\Lambda') \) if and only in \( \Lambda, \Lambda' \) are homothetic (which must be checked). Through the association \( \tau \mapsto \Lambda_\tau \) we can consider all of the functions above as function of \( \tau \). Then the homothety invariance of \( j \) is just saying that \( j(\tau) \) is invariant under the action of \( \text{SL}_2(\mathbb{Z}) \). In particular, \( j(\tau + 1) = j(\tau) \), since the is the action of \( T \) on \( \tau \), and hence there is a Fourier expansion in \( q = e^{2\pi i \tau} \):

\[
j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots
\]

The Fourier expansion shows that \( j \) is holomorphic everywhere, and has a simple pole at \( \infty \). Before moving on, we want to observe a few things about the functions above. For \( \gamma \in \text{SL}_2(\mathbb{Z}) \), we have that \( G_{2k}(\gamma \tau) = (c\tau + d)^{2k} G_{2k}(\tau) \). This is not invariant under \( \text{SL}_2(\mathbb{Z}) \), rather it has ‘weight ’ \( 2k \). The way we got an \( \text{SL}_2(\mathbb{Z}) \) invariant function was by taking quotients of functions of the same ‘weight’.

The above remark shows that \( j \) is well defined on the quotient \( H/\text{SL}_2(\mathbb{Z}) \). One may prove that \( j \) is surjective onto \( C \) and hence

\[
j: H/\text{SL}_2(\mathbb{Z}) \to C, \quad \tau \mapsto j(\Lambda_\tau)
\]

is a complex analytic isomorphism of open Riemann surfaces. Thus we can consider \( H/\text{SL}_2(\mathbb{Z}) \) as the \( Y(1)(C) \) for the affine curve \( Y(1) = \mathbb{A}^1 \) defined over \( Q \). We will want to work with compact Riemann surfaces, and we denote by \( X(1) \) the compactification of \( Y(1) \). From above we see that \( X(1)(C) \cong \mathbb{P}^1(C) \). With a view towards eventual generalizations, we would like to establish a more natural way of compactifying \( H/\text{SL}_2(\mathbb{Z}) \). Let \( H^* = H \cup \mathbb{P}^1(Q) \). So we are taking the upper half plane, adding the usual copy of the rational numbers in \( C \), as well as adding the point at infinity. Then \( \text{SL}_2(\mathbb{Z}) \) acts on \( H^* \), in fact transitively on \( \mathbb{P}^1(Q) \) and we define \( X(1) = \mathbb{P}^1 \) and see that \( X(1)(C) = H^*/\text{SL}_2(\mathbb{Z}) \). The complement of \( Y(1) \) in \( X(1) \) are called cusps, though in this case there is a single cusp. All of the above shows that elliptic curves over \( C \) are parameterized by the non cuspidal points complex points of a projective curve \( X(1) \) defined over \( Q \). In fancy language, we have constructed a coarse moduli space for elliptic curves. It is not a fine moduli space. In order to produce a fine moduli space, we need to add more data.
2. Modular Forms, Modular Functions and Congruence Subgroups

Consider congruence subgroups of SL$_2(\mathbb{Z})$:

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \mod N \right\};
\]

\[
\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \mod N, a \equiv b \equiv 1 \mod N \right\};
\]

\[
\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b \equiv c \equiv 0 \mod N, a \equiv b \equiv 1 \mod N \right\}.
\]

More generally, a congruence subgroup is a a subgroup of SL$_2(\mathbb{Z})$ that contains $\Gamma(N)$ for some $N$. Given a congruence subgroup $\Gamma$, we can form the quotient $H/\Gamma$ which can be given the structure of an open Riemann surface (as can it’s compactification $H^*/\Gamma$), much in the same way as $H/SL_2(\mathbb{Z})$. Since $\Gamma$ acts on $\mathbb{P}^1(\mathbb{Q})$ we can form the compactification $X(\Gamma)(\mathbb{C}) = H^*/\Gamma$ analogous to $X(1)(\mathbb{C})$. We should expect that for $\Gamma \neq SL_2(\mathbb{Z})$ that $H/\Gamma$ will have encode more data that $H/\Gamma$. We explore this for the congruence subgroup $\Gamma_1(N)$. Given any $\tau \in H/\Gamma_1(N)$ we can form the elliptic curve $E_\tau = C/\Lambda_\tau$. Consider the point $T_\tau \in E_\tau[N]$ corresponding to the point $\frac{1}{N} \in C/\Lambda_\tau$. We claim for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$, the point $T_{\gamma \tau}$ is fixed under the change of basis given by $\gamma$. Indeed under the isomorphism:

\[
C/\Lambda_\tau \rightarrow C/\Lambda_{\gamma \tau};
\]

\[
z \mapsto \frac{z}{c\tau + d}.
\]

$\frac{1}{N}$ maps to $\frac{1}{N(c\tau + d)}$. By assumption on $\gamma$ we have $c/N, (d-1)/N \in \mathbb{Z}$ and hence

\[
\frac{1}{N} - \frac{1}{N(c\tau + d)} = \frac{(c/N)\tau + (d-1)/N}{c\tau + d} \in f(\Lambda_\tau) = \Lambda_{\gamma \tau}
\]

Further, we see that $\gamma \in SL_2(\mathbb{Z})$ fixes $T_\tau$ if and only if $\gamma \in \Gamma_1(N)$. This suggests that $H/\Gamma_1(N)$ is classifying pairs $(E, T)$ with $E$ an elliptic curve and $T \in E$ a point of order $N$ up to isomorphism, where pairs $(E, T), (E', T')$ are isomorphic if there exists an isomorphism $E \rightarrow E'$ mapping $T$ to $T'$. This is indeed the case as one can show that every pair $(E, T)$ is isomorphic to $(E_{\tau}, T_{\tau})$ for some $\tau \in H/\Gamma_1(N)$. The spaces $H/\Gamma_0(N)$ and $H/\Gamma(N)$ admit similar interpretations: the former parameterizes pairs $(E, C)$ with $E$ an elliptic curve and $C$ a cyclic subgroup of order $N$, while the latter parameterizes pairs $(E, \{P, T\})$ where $E$ is an elliptic curve, and $\{P, T\}$ a basis of the $N$ torsion that pairs to $e^{2\pi i/N}$ under the Weil pairing.

As in the case of $SL_2(\mathbb{Z})$, we would like to say that $H^*/\Gamma$ is analytically equivalent to the complex points $X(\Gamma)(\mathbb{C})$ for some projective curve $X(\Gamma)$ defined over some subfield of $\mathbb{C}$ (hopefully a number field). This will indeed be the case, however to prove it, we need to construct enough functions on $H^*/\Gamma$. It is clear that meromorphic functions on $H/\Gamma$ are the same as meromorphic functions $f$ on $H$ such that $f(\gamma \tau) = f(\tau)$ for all $\tau \in H$. In order to find functions on $H$ invariant under $\Gamma$, it is easiest to find functions that transform a certain way under the action of $\Gamma$, and take quotients of these functions. This is analogous to how we find functions on projective space $\mathbb{P}^n(\mathbb{C})$: first we look at the homogenous forms $f$ on $\mathbb{C}^{n+1}$ satisfying $f(az) = a^n f(z)$ and get functions by taking quotients of homogenous forms of the same ‘weight’.

**Definition.** Let $\Gamma$ be a congruence subgroup. A meromorphic function $f$ on $H$ is called a modular function of weight $k$ for $\Gamma$ if

\[
(1) \quad f(\tau) = (c\tau + d)^{-k} f(\gamma \tau) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.
\]
(2) $f$ is meromorphic at the cusps of $H^*/\Gamma$.

If $f$ is holomorphic (including at the cusps) then $f$ is said to be a modular form of weight $k$. If further $f$ vanishes at the cusps, then $f$ is called a cusp form.

In the case $\Gamma = \text{SL}_2(\mathbb{Z})$ the condition at the cusps of $f$ can be phrased in terms of the Fourier expansion of $f$

$$f(\tau) = \sum_{n=0}^{\infty} c_n q^n,$$

(which exists since the first condition implies $f(\tau) = f(\tau + 1)$). A priori we could have $n_0 = \infty$. Condition (2) of the definition then says that $n_0$ is a finite integer. If $f$ is holomorphic on $H$, then $f$ is a modular form if $n_0 \geq 0$ and $f$ is a cusp form if $n_0 > 0$, i.e. $c_0 = 0$. We have already seen several examples of modular forms and functions

$$G_{2k}(\tau) = \sum_{a,b \neq 0} \frac{1}{(a+b\tau)^{2k}}, \quad \Delta(\tau), \quad j(\tau)$$

The Eisenstein series $G_{2k}(\tau)$ is a modular form of weight $2k$, $\Delta$ is a modular function of weight $12$, and $j$ is a modular function of weight $0$.

We list some immediate consequences. The modular forms of weight zero for $\Gamma$ are the holomorphic functions on $H^*/\Gamma$, hence are constant. The modular functions of weight $0$ are the meromorphic functions on $H^*/\Gamma$. Given a modular form $f$ of weight two, we note that $f(\tau)d\tau$ is an invariant differential form on $H$ (this follows from the identity $d\tau a\tau + b = \frac{ad-bc}{c\tau + d}d\tau$). Further, if $f$ is a cusp form then one shows that $f(\tau)d\tau$ is holomorphic at each of the cusps of $\Gamma$ and hence gives a holomorphic 1-form on $H^*/\Gamma$. This construction yields an isomorphism between the space of weight 2 cusp forms for $\Gamma$ and the space $H^0(H^*/\Gamma, \Omega^1)$ of holomorphic 1-forms on $H^*/\Gamma$. In fact, all modular forms of even weight can be constructed as sections of line bundles.

**Proposition 2.** The modular forms of weight $2k$ are the meromorphic sections of the line bundle $(\Omega^1)^{\otimes k}$.

**Proof.** This follows from the computation $\gamma^*dz = (cz + d)^{-2}dz$ for $\gamma = (z \mapsto \frac{az + b}{cz + d})$. 

One can also think of odd weight modular forms as appropriate sections of line bundles. Note that this agrees with how we think of homogenous forms on $\mathbb{C}^{n+1}$ as appropriate sections of line bundles on $\mathbb{P}^n$. And again similarly to this case, there are no modular forms of negative weight. Originally we were interested in modular forms to construct functions on $H/\Gamma$ for congruence subgroups, in order to show that $H/\Gamma$ are the complex points of a curve $Y(\Gamma)$. We will not prove this, however we state the important results.

**Theorem 1.** Let $N \geq 1$ be an integer.

1. There exists a smooth projective curve $X_0(N)$ and a complex analytic isomorphism

   $$j_{N,0}: H^*/\Gamma_0(N) \rightarrow X_0(N)(\mathbb{C})$$

   such that each $\tau \in H/\Gamma_0(N)$ corresponds to equivalence classes of pairs $(E,C)$ with $C$ a subgroup of order $N$. Further, this equivalence class contains a pair $(E,C)$ with $E$ and $C$ defined over the field $\mathbb{Q}(j_{N,0}(\tau))$.

2. There exists a smooth projective curve $X_1(N)$ and a complex analytic isomorphism

   $$j_{N,1}: H^*/\Gamma_1(N) \rightarrow X_1(N)(\mathbb{C})$$

   such that each $\tau \in H/\Gamma_1(N)$ corresponds to equivalence classes of pairs $(E,T)$ with $T$ a point of order $N$. Further, this equivalence class contains a pair $(E,T)$ with $E$ and $T$ defined over $\mathbb{Q}(j_{N,1}(\tau))$. 

(3) There exists a smooth projective curve \( X(N) \) and a complex analytic isomorphism

\[
j_N : \mathbb{H}^* / \Gamma(N) \to X(N)(\mathbb{C})
\]

such that each \( \tau \in \mathbb{H} / \Gamma(N) \) corresponds to equivalence classes of triples \((E, T_1, T_2)\) with \( T_1, T_2 \) generators of \( E[N] \) which pair to \( e^{2\pi i / N} \) under the Weil pairing. Further, each equivalence class admits a triple with \( E, T_1 \) and \( T_2 \) all defined over \( \mathbb{Q}(e^{2\pi i / N}, j_N(\tau)) \).

We end this section with an example. Suppose we want to know whether or not there exists an elliptic curve over \( \mathbb{Q} \) with a point of 11 torsion over \( \mathbb{Q} \). Remarkably, the non cuspidal points of \( X_1(11) \) form a fine moduli space for the moduli problem of classifying pairs \((E, T)\) with \( E \) an elliptic curve and \( T \) a point of order 11. In particular, there is an elliptic curve \( E \) over \( \mathbb{Q} \) with a point of 11 in \( E(\mathbb{Q}) \) if and only if there is a non cuspidal point \( X_1(11)(\mathbb{Q}) \). Thus we are reduced to computing rational points on a specific curve. Using modular forms, one can show that the equation of \( X_1(11) \) is

\[
y^2 + y = x^3 - x
\]

It can be shown that there are only 5 rational points on this curve: \( \{O, (0,0), (1,-1), (0,-1), (1,0)\} \). Further, one can show that these are all cusps, and hence there are no elliptic curves \( E \) over \( \mathbb{Q} \) with a point of order 11 in \( E(\mathbb{Q}) \).

3. Important Conjectures

Modular curves and modular forms arise frequently in arithmetic geometry. An elliptic curve \( E \) over \( \mathbb{Q} \) is a modular elliptic curve if there is a finite covering \( X_0(N) \to E \). One should keep in mind that this concept of a modular elliptic curve is different from a modular curve as defined in the last section that happens to be elliptic. The following theorem was known as the Taniyama-Shimura conjecture.

Theorem 2. (Modularity Theorem) Every elliptic curve over \( \mathbb{Q} \) is a modular elliptic curve.

The proof for \( E \) semistable was given by Andrew Wiles in 1995, with the help of Richard Taylor. The full theorem was proven later. Work of Serre, Frey and Ribet earlier had shown that this case of the modularity theorem was enough to prove Fermat’s last theorem. Very roughly, previous work had shown that given a nontrivial solution of the Fermat equation \( x^n + y^n = z^n \), one could produce an elliptic curve (the Frey curve) which is semistable but not modular. So modular curves played a key role in one of the greatest theorems in mathematics.

Modular curves and modular forms play a key role in the formulation of the Birch and Swinnerton-Dyer conjecture. In order to formulate this conjecture, we have to describe the notion of the \( L \)-series of an elliptic curve \( E \) over a number field \( K \). The \( L \)-series is defined as

\[
L(E/K, s) = \prod_p \frac{1}{L_p(q_p^{-s})}
\]

ranging over primes \( p \) of \( \mathcal{O}_K \), the ring of integers of \( K \). The term \( q_p \) is \( |\kappa(p)| \), the size of the residue field \( \kappa(p) = \mathcal{O}_K / p \); and the \( L_p \) are the local \( L \)-functions. In order to defined the local \( L \)-functions, let \( E_p \) be the reduction of \( E \) mod \( p \) and define \( a_p = q_p + 1 - |E_p(\kappa(p))| \). Then the local \( L \)-functions are

\[
L_p(T) = \begin{cases} 
1 - a_p T + q_p T^2 & \text{if } E \text{ has good reduction at } p \\
1 - T & \text{if } E \text{ has nodal reduction at } p \text{ with rational tangents} \\
1 + T & \text{if } E \text{ has nodal reduction at } p \text{ with irrational tangents} \\
1 & \text{if } E \text{ has cuspidal reduction at } p
\end{cases}
\]

Now all of this is fairly frightening, but the important thing to notice is that the \( L \)-series is defined in terms of local data. That is, it keeps track of the behavior – singularity and number of points – of
the reduction of $E$ at primes. The $L$-series $L(E/K, s)$ is known to converge to an analytic function in the right half plane $\text{Re}(s) > \frac{3}{2}$. However we have the following theorem.

**Theorem 3.** For any elliptic curve $E$ over $K$, the $L$-series $L(E/K, s)$ has an analytic continuation to the entire complex plane.

This theorem ends up being a corollary of the modularity theorem, by relating $L(E/K, s)$ to modular forms, when $E$ is a modular elliptic curve. Applications like these are one of the reasons that modular forms are so important. The above theorem allows us to pose the million dollar question.

**Conjecture 1.** (Birch and Swinnerton-Dyer) Let $E$ be an elliptic curve over a number field $K$. The order of the zero of $L(E/K, s)$ at $s = 1$ is equal to the rank of the abelian group $E(K)$.

This remarkable conjecture is somehow relating the local properties of an elliptic curve to global properties.

I’ll end these notes by just mentioning that modular curves are the simplest interesting examples of Shimura varieties. Shimura varieties are extremely important in arithmetic geometry, though they are quite hard to define. One can think about Shimura varieties in several ways:

1. Shimura varieties are highly symmetric objects with rich actions of Lie groups and their discrete subgroups.
2. Shimura varieties are moduli spaces for abelian varieties possibly with extra structure.
3. Shimura varieties are moduli spaces for Hodge structures.

Our approach to modular curves illustrates the first two examples. We have already seen how modular curves $X(\Gamma)$ are moduli space of elliptic curves (dimension 1 abelian varieties), perhaps with extra structure. However, we also gave a construction in terms of a quotient $H^*/\Gamma$. The Hermitian symmetric space $H$ has an action of $\text{SL}_2(\mathbb{R})$, and we are quotienting by the discrete subgroup $\text{SL}_2(\mathbb{Z})$. The last interpretation is made by the equivalence of abelian varieties and integral Hodge structures of weight 1. Of course all of this is very vague because I know very, very little about Shimura varieties.

**References**