The Universal Double Point Formula

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These notes are for a talk in the Student Algebraic Geometry Seminar. In order to get to the Double Point Formula, many definitions will be imprecise. The base manifold X is always be assumed to be smooth, complex, connected, and compact (think of X as a projective variety if you prefer). All vector bundles on X are complex (hence they are non-compact complex manifolds), all dimensions and ranks are complex, and all maps are assumed to be holomorphic. These conventions are for convenience; the theory can be developed in greater generality.

1 Vector bundles

1.1 Definition and first examples

Let X be a manifold of dimension n. A *vector bundle* $\mathcal{V}$ *over* X *of rank* r is a manifold $\mathcal{V}$ with a map $\pi\colon \mathcal{V} \to X$ whose fibers $\mathcal{V}_p$ for $p \in X$ are r-dimensional $\mathbb{C}$-vector spaces.

Example 1.1. Every X has trivial bundles $X \times \mathbb{C}^r$ for all $r \geq 0$ and a tangent bundle T_X. If $f\colon X \to Y$ is a map and $\mathcal{V}$ is a vector bundle on Y, there is a pull back vector bundle $f^*\mathcal{V}$ on X whose fiber at a point $p \in X$ is the same as the fiber at $f(p) \in Y$ of $\mathcal{V}$.

Many operations that can be performed on vector spaces to produce new vector spaces can also be used fiberwise on vector bundles: direct sum $\oplus$, tensor product $\otimes$, dualizing $^\vee$, $\underline{\mathrm{Hom}}$ functors, short exact sequences, etc.

Example 1.2. Our main focus will be on the manifold $\mathbb{P}^n$, which parametrizes lines through the origin in $\mathbb{C}^{n+1}$. We will write points in $\mathbb{P}^n$ as $(n+1)$-tuples of complex numbers $(x_0 : x_1 : \cdots : x_n)$ up to scaling by nonzero numbers or as $[L]$, where L is the line spanned by $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$. $\mathbb{P}^n$ has a universal line bundle $\mathcal{S}$ (whose fibers are the lines being parametrized) and a universal rank n quotient bundle $\mathcal{Q}$ that fit in an exact sequence of vector bundles

$$0 \to \mathcal{S} \xrightarrow{\alpha} \mathbb{P}^n \times \mathbb{C}^{n+1} \xrightarrow{\beta} \mathcal{Q} \to 0.$$

The tangent bundle to $\mathbb{P}^n$ is $\underline{\mathrm{Hom}}(\mathcal{S}, \mathcal{Q}) \cong \mathcal{S}^\vee \otimes \mathcal{Q}$. This can be seen geometrically. A tangent vector at a point of $\mathbb{P}^n$ corresponding to a line $L \subset \mathbb{C}^{n+1}$ should be a direction in which that line can be perturbed, which corresponds to a map $L \to \mathbb{C}^{n+1}/L$.

1.2 Sections of vector bundles

A *section* of a vector bundle $\pi\colon \mathcal{V} \to X$ is a map $\sigma\colon X \to \mathcal{V}$ such that $\pi \circ \sigma = \mathrm{id}_X$. Namely, σ picks a vector in each fiber of the vector bundle. The collection of all sections of $\mathcal{V}$ forms a finite-dimensional $\mathbb{C}$-vector space $\Gamma(\mathcal{V})$. The *zeros* $Z(\sigma)$ of a section σ are the points $p \in X$ for which $\sigma(p)$ is the zero vector in $\mathcal{V}_p$. We say $\mathcal{V}$ is *globally generated* if the sections of $\mathcal{V}$ span $\mathcal{V}_p$ at each point.

Example 1.3. Every $\mathcal{V}$ has the zero section. The sections of the trivial bundle $X \times \mathbb{C}^n$ are isomorphic to $\mathbb{C}^n$. (Composing σ with the projection to $\mathbb{C}^n$ gives a holomorphic map from a compact manifold to $\mathbb{C}^n$, which must be constant by the open mapping theorem.)

Example 1.4. The sections of the trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$ (which correspond to fixing a vector $v \in \mathbb{C}^{n+1}$) induce sections of $\mathcal{Q}$ by composing with β; in fact, $\Gamma(\mathbb{P}^n \times \mathbb{C}^{n+1}) \cong \Gamma(\mathcal{Q})$, and both of these vector bundles are globally generated. The section σ_v on $\mathcal{Q}$ vanishes at a point $[L] \in \mathbb{P}^n$ exactly when $v \in L$. Thus $Z(\sigma_v) = \{[L_{0v}]\}$ is a single point of $\mathbb{P}^n$ corresponding to the unique line through the origin and v as long as $v \neq 0$.

2 Cohomology and Chern classes

If you haven't seen the *cohomology ring* $H^*(X, \mathbb{C})$, imagine that it is a vector space generated by classes $[Y]$ of submanifolds $Y \subset X$, where two such classes are equivalent if one can be deformed to get the other. The multiplication, called *cup product*, can be though of as intersecting submanifolds (after deforming one to ensure they intersect transversally). The multiplicative unit, which we will denote by 1, is equal to $[X]$. All points are equivalent and we denote the class by ρ. The cohomology ring has a grading ranging from degree 0 (multiples of 1) to degree $2n$ (multiples of ρ) corresponding to twice the (complex) codimension of the submanifolds, and the multiplication respects this grading.

A map $f\colon X \to Y$ induces linear maps $f_*\colon H^*(X, \mathbb{C}) \to H^*(Y, \mathbb{C})$ and $f^*\colon H^*(Y, \mathbb{C}) \to H^*(X, \mathbb{C})$ by taking images or preimages of submanifolds. The map f^* is even a morphism of rings that respects the grading. For example, $[Y] = f_* 1$ under the inclusion map $f\colon Y \hookrightarrow X$.

Example 2.1. $H^*(\mathbb{P}^n, \mathbb{C}) \cong \mathbb{C}[H]/H^{n+1}$, where $H \in H^2(X, \mathbb{C})$ is the class of a hyperplane. H^k is the class of a codimension k linear space, and in particular $\rho = H^n$. A submanifold Y of codimension m has degree d if $[Y].H^{n-m} = d\rho$, namely $[Y] = dH^m$.

The *kth Chern class* $c_k(\mathcal{V})$ of a globally generated vector bundle $\mathcal{V}$ of rank r is the class in $H^{2k}(\mathcal{V}, \mathbb{C})$ of the locus where $r - k + 1$ general sections of $\mathcal{V}$ are dependent. Note that $c_0(\mathcal{V}) = 1$ and $c_k(\mathcal{V}) = 0$ if $k > r$ or if $k > n$. The Chern classes are functorial with respect to pull back, namely if $f\colon X \to Y$ is a map and $\mathcal{V}$ is a vector bundle on Y, then $c_k(f^*\mathcal{V}) = f^*(c_k(\mathcal{V}))$.

Example 2.2. We saw that every non-zero section of the universal quotient bundle $\mathcal{Q}$ on $\mathbb{P}^n$, which is globally generated, vanishes at a single point. Thus $c_n(\mathcal{Q}) = \rho$. Moreover, n general sections of $\mathcal{Q}$ (corresponding to n vectors in $\mathbb{C}^{n+1}$) drop rank at a line $[L]$ exactly when L is contained in their span. Since the set of lines contained in a particular hyperplane is a hyperplane in $\mathbb{P}^n$ (a copy of $\mathbb{P}^{n-1}$), we see that $c_1(\mathcal{Q}) = H$.

The *total Chern class* $c(\mathcal{V})$ is defined to be $1 + c_1(\mathcal{V}) + \cdots + c_r(\mathcal{V}) \in H^*(X, \mathbb{C})$. It is multiplicative on exact sequences: if $0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to \mathcal{V}_3 \to 0$ is a short exact sequence of vector bundles, then $c(\mathcal{V}_1)c(\mathcal{V}_3) = c(\mathcal{V}_2)$. This multiplicative property can be used to define Chern classes for non-globally generated vector bundles. Dualizing a vector bundle just changes the sign of the odd Chern classes.

Example 2.3. $c(X \times \mathbb{C}^r) = 1$ since globally independent sections of a trivial bundle are independent at each fiber.

Example 2.4. We compute $c(T_{\mathbb{P}^n})$, recalling that $T_{\mathbb{P}^n} \cong \mathcal{S}^\vee \otimes \mathcal{Q}$. The universal short exact sequence

$$0 \to \mathcal{S} \to \mathbb{P}^n \times \mathbb{C}^{n+1} \to \mathcal{Q} \to 0$$

yields the identity $c(\mathcal{S})c(\mathcal{Q}) = 1$, namely

$$(1 + c_1(\mathcal{S}))(1 + H + c_2(\mathcal{Q}) + \cdots + \rho) = 1,$$

which implies $c_1(\mathcal{S}) = -H$. Now, tensoring the universal sequence by $\mathcal{S}^\vee$ yields

$$0 \to \mathbb{P}^n \times \mathbb{C} \to (\mathcal{S}^\vee)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0,$$

so

$$c(T_{\mathbb{P}^n}) = c(\mathcal{S}^\vee)^{n+1} = (1 + H)^{n+1}.$$

Note that the top term H^{n+1} of this expression vanishes because its degree is too large.

3 The Universal Double Point Formula

Let $f\colon X \to Y$ be a map with $\ell := \dim Y - \dim X > 0$. A point $q \in Y$ is a *double point of f* if q has at exactly two preimages. The set of all double points of f is denoted Y_2.

Example 3.1. The image of an immersion $f\colon \mathbb{P}^1 \to \mathbb{P}^2$ of degree d is a rational nodal plane curve of degree d. The double points of f are the (simple) nodes of the image. Assume all nodes are simple. A degree d plane curve has arithmetic genus $\binom{d-1}{2}$, which for a nodal curve equals the topological genus of the curve plus the number of nodes. Since the curve is rational (genus 0), it must have $\binom{d-1}{2}$ nodes.

We can compute the cohomology class of the closure of the set of double points using an extremely general formula involving the Chern classes of the tangent bundles of X and Y.

Theorem 3.2 (Universal Double Point Formula [Ron80])**.** *Suppose Y_2 has codimension 2ℓ and the loci of other singularities have codimension $> 2\ell$. Then the class of the closure of the set of double points is*

$$[\overline{Y_2}] = \tfrac{1}{2}\big((f_*1)^2 - f_*c_\ell(f)\big) \in H^{4\ell}(Y, \mathbb{C}),$$

where $c_\ell(f) = c_\ell(f^*T_Y/T_X)$.

We will say more about the "other singularities" later. Here f_*1 is the cohomology class of the image of f. There is an inclusion of vector bundles $T_X \to f^*T_Y$, and the quotient f^*T_Y/T_X is the (virtual) normal bundle of X in Y. Using multiplicativity, its total Chern class is

$$c(f) = 1 + c_1(f) + c_2(f) + \cdots + c_\ell(f) = \frac{1 + f^*c_1(T_Y) + \cdots + f^*c_{\dim Y}(T_Y)}{1 + c_1(T_X) + \cdots + c_{\dim X}(T_X)}.$$

This looks scarier than it is, as we will see in the next example.

Example 3.3. Let's compute the double point formula in the case of nodal degree d plane curves, where we already know the answer should be $\binom{d-1}{2}\rho$. We first note that f_*1 is the class of the image of f, which is dH since f has degree d. Next, we recall that $c(T_{\mathbb{P}^2}) = (1+H)^3$ and that $c(T_{\mathbb{P}^1}) = (1+\rho)^2$. Since f has degree d, $f^*H = d\rho$. Thus

$$c(f^*T_{\mathbb{P}^2}/T_{\mathbb{P}^1}) = \frac{(1+d\rho)^3}{(1+\rho)^2} = \frac{1+3d\rho}{1+2\rho} = (1+3d\rho)(1-2\rho) = 1+(3d-2)\rho,$$

so $c_1(f^*T_{\mathbb{P}^2}/T_{\mathbb{P}^1}) = (3d-2)\rho$. Putting everything together,

$$[(\mathbb{P}^2)_2] = \tfrac{1}{2}\big((dH)^2 - (3d-2)\rho\big) = \binom{d-1}{2}\rho.$$

Remark 3.4. When $2\ell = \dim Y$, there should be finitely many double points, and the condition on the codimension of other singularities is that these other singularities do not occur. When f is a degree d map $\mathbb{P}^1 \to \mathbb{P}^2$, this means that f is an immersion and its image has only simple nodes. To see what can go wrong if f has a cusp, consider the rational cuspidal cubic curve, which is the image of the map $f\colon \mathbb{P}^1 \to \mathbb{P}^2$ defined by $(x_0 : x_1) \mapsto (x_0^3 : x_0^2x_1 : x_1^3)$. The map f is a bijection, so there are no double points. Yet the double point formula is identical to the case of the degree 3 rational nodal curve, which had $\binom{3-1}{2} = 1$ double point. The problem is that the cusp, which is a "worse singularity" than a double point (the expected codimension of the locus of cuspidal points is $2\ell+1 = 3$), gets counted as a double point by mistake.

Example 3.5. For $d \geq 1$, let $f\colon \mathbb{P}^2 \to \mathbb{P}^4$ be defined by $(x_0 : x_1 : x_2) \mapsto (m_0 : m_1 : m_2 : m_3 : m_4)$, where the m_i are general degree d monomials in the x_i. (For $d \geq 2$, one can think of f as the composition of the d-uple embedding $f_d\colon \mathbb{P}^2 \to \mathbb{P}^{\binom{d+2}{2}-1}$ with a general projection from $\binom{d+2}{2} - 5$ points.) Then $f^*H = dh$ since the vanishing of a coordinate in $\mathbb{P}^4$ is the vanishing of the corresponding degree d monomial on $\mathbb{P}^2$. By the projection formula, $(f_*1)H^2 = f_*(f^*H^2) = d^2\rho$, so the image of f has class d^2H^2. Next, we compute

$$\begin{aligned} c(f^*T_{\mathbb{P}^4}/T_{\mathbb{P}^2}) = \frac{(1+dh)^5}{(1+h)^3} = \frac{1+5dh+10d^2\rho}{1+3h+3\rho} &= (1+5dh+10d^2\rho)(1-3h+6\rho) \\ &= 1+(5d-3)h+(10d^2-15d+6)\rho, \end{aligned}$$

in particular $c_2(f) = (10d^2-15d+6)\rho$. Thus the double point formula yields

$$[(\mathbb{P}^4)_2] = \tfrac{1}{2}((d^2H^2)^2 - (10d^2-15d+6)\rho) = \tfrac{1}{2}(d-1)(d-2)(d^2+3d-3)\rho.$$

In particular, $d = 1$ and $d = 2$ yield 0 double points (in these cases f is an embedding; this is obvious when $d = 1$ and happens for $d = 2$ because the secant variety of the 2-uple embedding fails to fill $\mathbb{P}^5$!), $d = 3$ yields 15, $d = 4$ yields 75, $d = 5$ yields 222, $d = 6$ yields 510, and so on.

Example 3.6. For $d \geq 2$, let $f\colon \mathbb{P}^2 \to \mathbb{P}^3$ be defined as in the previous example but with only four general degree d monomials. We still have $f^*H = dh$, from which we compute $f_*1 = d^2H$ and $f_*h = dH^2$. Moreover,

$$c(f^*T_{\mathbb{P}^3}/T_{\mathbb{P}^2}) = \frac{(1+dh)^4}{(1+h)^3} = (1+4dh+6d^2\rho)(1-3h+6\rho) = 1+(4d-3)h+(6d^2-12d+6)\rho,$$

and class we need is $c_1(f) = (4d-3)h$. The closure of the double point locus is a curve (containing finitely many triple points and finitely many pinch points in addition to double points), whose class is

$$[(\mathbb{P}^3)]_2 = \frac{1}{2}((d^2H)^2 - d(4d-3)H^2) = \frac{1}{2}d(d^3 - 4d + 3)H^2.$$

Remark 3.7. In the literature, a "singularity" is a point at which the derivative drops rank, while a fiber consisting of more than one point is a "multisingularity". Cuspidal ("pinch") points are the simplest singularities, while double points are the simplest multisingularities. Universal formulas are known to exist for all types of singularities (these are called Thom polynomials, though few have been computed) and for some multisingularities (such as double points). See [Ron80] for multiple point formulas for immersions, [Kle81] for multiple point formulas for maps of corank 1, and [Kaz03] for conjectural formulas for general multisingularities (supported by topological arguments).

References

[Kaz03] M. E. Kazarian. Multisingularities, cobordisms, and enumerative geometry. *Russian Mathematical Surveys*, 58(4):665–724, 2003.

[Kle81] Steven L. Kleiman. Multiple-point formulas I: Iteration. *Acta Mathematica*, 147(1):13–49, December 1981.

[Ron80] Felice Ronga. On multiple points of smooth immersions. *Commentarii Mathematici Helvetici*, 55:521–527, 1980.