ADJUNCTION AND INVERSION OF ADJUNCTION

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Abstract. These notes are meant to be a handout for the student seminar in algebraic geometry at the University of Utah. They are a short introduction to the notions of adjunction and inversion of adjunction for pairs.

Contents

1. Introduction

Among the techniques in birational geometry, adjunction theory is one of the most powerful tools. It allows to relate the geometry, and in particular the singularities, of the ambient variety with the one of appropriate subvarieites. We call *adjunction* the process of inferring statements about a subvariety from some knowledge of the ambient variety, while the inverse and usually more complicated process is called *inversion of adjunction*. The most satisfactory formulation of this theory in the case of pairs is the following, due to Hacon [Hac14].

Theorem 1.1 ([Hac14, Theorem 0.1])**.** *Let V be a log canonical center of a pair $(X, \Delta = \sum \delta_i \Delta_i)$ where $0 \leq \delta_i \leq 1$. Then (X, Δ) is log canonical in a neighborhood of V if and only if $(V, \mathbf{B}(V; X, \Delta))$ is log canonical.*

In the case V has codimension 1, the statement takes the following and simpler form, originally due to Kawakita [Kaw07].

Theorem 1.2 ([Kaw07])**.** *Let $(X, S + B)$ be a log pair such that S is a reduced divisor which has no common component with the support of B, let S^ν denote the normalization of S, and let B^ν denote the different of B on S^ν. Then $(X, S + B)$ is log canonical near S if and only if (S^ν, B^ν) is log canonical.*

The purpose of this note is to give an overview of the ideas behind these results, and to present a slightly simplified version of Theorem 1.2.

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2. Adjunction on singular varieties: the different

Consider a smooth projective variety X, and let D be a smooth prime divisor on it. Then, the classic literature gives an explicit relation between the line bundles $\omega_X = \mathcal{O}_X(K_X)$ and $\omega_D = \mathcal{O}_D(K_D)$ [Har77, Proposition II.8.20]. The *adjunction formula* tells us that

$$\omega_X(D)|_D \cong \omega_D.$$

The above can be rephrased as

$$(K_X + D)|_D = K_D$$

in the language of divisors.

A natural question is how much such a relation can be generalized in the case of singular varieties. We will explore this through an example.

Let X be the cone over a projectively normal rational curve of degree n, and let L be a line through the vertex V. If we blow up V, we obtain a morphism $\pi : \tilde{X} \to X$, where $\tilde{X}$ is smooth. Denote by $\tilde{L}$ the strict transform of L, and by E the π-exceptional divisor.

A direct computation shows that K_X and L are a $\mathbb{Q}$-Cartier divisor with the latter having Cartier index exactly n [Har77, cf. Example II.6.5.2]. Also, one can show that $E^2 = -n$.

What happens in the smooth case suggests to consider the divisor $(K_X+L)|_L$. As K_X+L is just a $\mathbb{Q}$-Cartier divisor, the direct computation is not the most convenient approach. Instead, we notice that the morphism $\pi_{\tilde{L}} : \tilde{L} \to L$ obtained restricting π is an isomorphism. Under the identification given by $\pi_{\tilde{L}}$, we have

$$(K_X + L)|_L = (\pi^*(K_X + L))|_{\tilde{L}}.$$

As π is birational, we have $K_Y = \pi^* K_X + aE$ for some a. The adjunction formula for smooth varieties gives us

$$K_E = (K_Y + E)|_E = (\pi^* K_X + (a+1)E)|_E.$$

As E is a rational curve, this gives

$$-2 = \deg(\pi^* K_X + (a+1)E)|_E = (\pi^* K_X + (a+1)E) \cdot E = -n(a+1).$$

Thus, we have $a = -1 + \frac{2}{n}$.

By construction, $nL \sim H$, where H is a very ample divisor with $H^2 = n$. In particular, this implies $L^2 = \frac{1}{n}$. Now, consider $\pi^* L = \tilde{L} + bE$. Then, we have

$$\frac{1}{n} = L^2 = (\pi^* L)^2 = \tilde{L}^2 + 2b\tilde{L} \cdot E + E^2.$$

As $\tilde{L}^2 = 0$ and $\tilde{L} \cdot E = 1$, we conclude that $b = \frac{1}{n}$.

Putting all the above together, we get

$$(K_X + L)|_L = \left(K_Y + \tilde{L} + \left(1 - \frac{1}{n}\right) E \right)\Big|_{\tilde{L}} = K_{\tilde{L}} + \left(1 - \frac{1}{n}\right) O,$$

where $\{0\} \coloneqq \tilde{L} \cap E$ is identified with V via $\pi_{\tilde{L}}$. We will call the correction term appearing in the formula *different*, and we will write

$$\mathrm{Diff}(0) \coloneqq \left(1 - \frac{1}{n}\right) V.$$

Such a term appears because of the singularities of X, and the fact that L is not a Cartier divisor.

This example illustrates how in birational geometry it is natural to consider the notion of *pair* (X, Δ), where X is a normal projective variety, and Δ is an effective $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Indeed, on a mildly singular variety as the cone over a rational curve, the natural operation of adjunction produces a pair with non-zero boundary. This is a common phenomenon, as many proofs in higher dimensional geometry rely on induction on dimension and adjunction. Therefore, even if we are interested in statement A for varieties, the inductive step would require having knowledge of statement A for pairs in lower dimension.

Now, assume that (X, Δ) is a pair, where $\Delta = S + B$, with S normal prime divisor. Let $\pi : Y \to X$ a *log resolution* of (X, Δ), i.e. a birational morphism with Y smooth, and $\mathrm{Supp}(\pi_*^{-1}\Delta) + \mathrm{Ex}(\pi)$ is a simple normal crossing divisor. Denote by $\tilde{S}$ the strict transform of S. Then, we can write

$$K_Y + \Delta_Y = \pi^*(K_X + \Delta),$$

and then define

$$\mathrm{Diff}(B) := \pi_{\tilde{S},*}((\Delta_Y - \tilde{S})|_{\tilde{S}}),$$

where $\pi_{\tilde{S}} : \tilde{S} \to S$ is the restriction of π. One can show that the definition does not depend on Y, and that $\mathrm{Diff}(B)$ is effective [Hac14]. Thus, this procedure provides us with a new pair $(S, \mathrm{Diff}(B))$.

3. A quick review of singularities of pairs

Given a pair (X, Δ), it is natural to introduce some measure of its singularities. As $K_X + \Delta$ is $\mathbb{Q}$-Cartier, for any birational morphism $\pi : Y \to X$ from a normal variety, we can consider

$$K_Y + \Delta_Y = \pi^*(K_X + \Delta),$$

where Δ_Y is implicitly defined, given the choice $\pi_* K_Y = K_X$.

If X is smooth and $\Delta = 0$, then $\Delta_Y < 0$ for all $Y \neq X$. On the other hand, as we saw with the cone over a rational curve, this is not always the case for singular varieties. Furthermore, if $\Delta \neq 0$, the bigger its coefficients are, the more positive Δ_Y becomes. In particular, if Δ has some irreducible component with coefficient greater than one, one can find a higher model Y where some coefficient of Δ_Y gets arbitrarily positive [KM98, cf. Chapter 2].

Therefore, the coefficients of Δ_Y, as Y varies among the higher birational models of X, measure how singular the pair (X, Δ) is. We say that (X, Δ) is *Kawamata log terminal*, in short klt, if for any divisor E on a model Y we have $\mathrm{mult}_E(\Delta_Y) < 1$. Similarly, we say that (X, Δ) is *log canonical*, in short lc, if $\mathrm{mult}_E(\Delta_Y) \leq 1$ for any such E. Log canonical singularities are the broader class of singularities appearing in the minimal model program. Kawamata log terminal singularities represent a class of milder singularities, and they are considerably better behaved than general log canonical ones.

Given a pair (X, Δ), there is an algorithmic way to determine whether it is klt or lc. Let $\pi : Y \to X$ be a log resolution of (X, Δ). Then, (X, Δ) is klt if and only if $\mathrm{mult}_E(\Delta_Y) < 1$ for any prime divisor E on Y. Similarly, (X, Δ) is lc if and only if $\mathrm{mult}_E(\Delta_Y) \leq 1$ for any such E [KM98, cf. Lemma 2.30 and Corollary 2.31].

As our main focus is adjunction, we are interested in pairs (X, Δ) with $\Delta = S + B$, and S normal prime divisor. Clearly, (X, Δ) is not klt, as $\mathrm{mult}_S(\Delta) = 1$. Still, we can ask for it to be almost klt. If a pair (X, Δ) is log canonical, and, in addition, $\mathrm{mult}_E(\Delta_Y) < 1$ for all Y and E exceptional over X, we say that (X, Δ) is *purely log terminal*, in short plt. Thus, a pair (X, Δ) is plt if the multiplicities of the exceptional divisors behave as in the klt case, while Δ is allowed to have components with coefficient 1.

4. Adjunction and singularities

In the previous sections we introduced the notion of different, and discussed a way of measuring the singularities of a pair. Given a pair $(X, \Delta = S + B)$, the adjunction formula provides a new pair $(S, \mathrm{Diff}(B))$. Therefore, it is natural to investigate how the singularities of these two relate.

Let $\pi : Y \to X$ be a log resolution of the pair $(X, S + B)$, and denote by $\pi_{\tilde{S}} : \tilde{S} \to S$ the induced morphism from the strict transform of S to S itself. As S is part of the boundary, $\tilde{S}$ is smooth, and $\mathrm{Ex}(\pi_{\tilde{S}}) = \mathrm{Ex}(\pi) \cap \tilde{S}$. Thus, as $\tilde{S} + \mathrm{Ex}(\pi)$ is a simple normal crossing divisor, $\pi_{\tilde{S}}$ is a log resolution of $(S, \mathrm{Diff}(B))$.

Now, we want to rephrase what it means for $(X, S + B)$ and $(S, \mathrm{Diff}(B))$ to have a certain kind of singularities in terms of the coefficients of the π-exceptional divisors. By further blow-ups, we may assume that every π-exceptional divisor intersecting $\tilde{S}$ has center contained in S. Then, write

$$\Delta_Y = \sum_i a_i D_i,$$

and define divisors

$$A \coloneqq \sum_{i | a_i < 1} a_i D_i, \quad F \coloneqq \sum_{i | a_i \geq 1} a_i D_i.$$

First, we notice that $(X, S + B)$ is log canonical if and only if $F = \lfloor F \rfloor$, and it is plt if and only if it is log canonical, F is not exceptional, and it is the disjoint union of its irreducible components [KM98, cf. Corollary 2.31 and Proposition 5.51].

Now, define $\tilde{F} \coloneqq F - \tilde{S}$. Then, $(S, \mathrm{Diff}(B))$ is klt if and only if $\tilde{S} \cap \tilde{F} = \emptyset$. Similarly, $(S, \mathrm{Diff}(B))$ is log canonical if and only if $\tilde{F} = \lfloor \tilde{F} \rfloor$ holds in a neighborhood of $\tilde{S}$.

Our goal is to relate the singularities of $(X, S + B)$ and the ones of $(S, \mathrm{Diff}(B))$. Clearly, we can not infer anything about the geometry of $(X, S + B)$ away from S just looking at $(S, \mathrm{Diff}(B))$. Therefore, we will more precisely be interested into relating the singularities of $(X, S + B)$ in a neighborhood of S and the ones of $(S, \mathrm{Diff}(B))$.

Therefore, we have to refine the above observations in order to talk about the singularities of $(X, S + B)$ in a neighborhood of S. The divisors contributing to the singularities in a neighborhood of S are the divisors on Y that map to S. Equivalently, they are the divisors on Y that intersect $\pi^{-1}(S)$. Hence, $(X, S + B)$ is plt in a neighborhood of S if and only if $\pi^{-1}(S) \cap F = \emptyset$. Analogously, $(X, S + B)$ is log canonical near S if and only if $F = \lfloor F \rfloor$ in a neighborhood of $\pi^{-1}(S)$.

These observations lead to the following statement, also known as *adjunction* [KM98, cf. Proposition 5.46].

Theorem 4.1. *Let $(X, S+B)$ be a pair, where S is a normal prime divisor. If $(X, S+B)$ is plt near S, then $(S, \mathrm{Diff}(B))$ is klt. Analogously, if $(X, S+B)$ is log canonical near S, then $(S, \mathrm{Diff}(B))$ is log canonical.*

From the above discussion, it is not clear whether there is any way to reverse the implications in Theorem 4.1. Indeed, imagine there are two exceptional divisors E_1 and E_2 that are both mapped into S by π, and assume that $E_1 \cap \tilde{S} \neq \emptyset$ and $E_2 \cap \tilde{S} = \emptyset$. As $E_1 \cup E_2 \subset \pi^{-1}(S)$, we need to control the coefficients of both of them to infer anything about the singularities of $(X, S+B)$ near S. On the other hand, only E_1 contributes to the singularities of $(S, \mathrm{Diff}(B))$, while E_2 is apparently irrelevant.

Surprisingly, we have the following very powerful result.

Theorem 4.2 ([KM98, Corollary 5.49])**.** *Let $g : Y \to X$ be a proper and birational morphism and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor such that $g_* D$ is effective and $-(K_Y + D)$ is g-nef.*

Let $Z \subset Y$ be the subset of points where (Y, D) is not sub-klt. Then Z is connected in a neighborhood of any fiber of g.

Theorem 4.2 is a direct consequence of the *connectedness principle* [KM98, Theorem 5.48]. As in the statement D is not necessarily effective, we say it is a sub-boundary. Thus, what is called klt in the case of pairs, is named sub-klt in such a setup.

Now, we apply Theorem 4.2 to the above situation. The sub-boundary D will be Δ_Y. Thus, $\pi_* \Delta_Y = \Delta \geq 0$ is a boundary, and, as $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$, $-(K_Y + \Delta_Y)$ is π-trivial. Hence, the hypotheses of Theorem 4.2 are satisfied. Then, assume $(S, \mathrm{Diff}(B))$ is klt. For every $x \in S$ there is an open neighborhood U_x such that $\pi^{-1}(U_x) \cap (\tilde{S} \cup \tilde{F})$ is connected. As $\tilde{S} \cap \tilde{F} = \emptyset$, we conclude that $\pi^{-1}(U_x) \cap \tilde{F} = \emptyset$. As x varies in S, this implies that $(X, S+B)$ is plt.

Thus, we get a first formualtion of *inversion of adjunction* [KM98, cf. Theorem 5.50].

Theorem 4.3. *Let $(X, S+B)$ be a pair, where S is a normal prime divisor. Then, $(X, S+B)$ is plt near S if and only if $(S, \mathrm{Diff}(B))$ is klt.*

Unfortunately, a pair (X, Δ) may or may not be log canonical along the locus where it fails to be Kawamata log terminal. Therefore, Theorem 4.2 can not be applied to get a full statement for inversion of adjunction in the log canonical case. On the other hand, it can still be exploited to get some partial result in this direction [KM98, cf. Theorem 5.50].

5. Log canonical inversion of adjunction

As discussed in the previous section, it is natural to expect inversion of adjunction to hold also in the log canonical case. Unfortunately, the techniques illustrated so far are not powerful enough to perform such extension. Using more modern techniques, Kawakita proved inversion of adjunction in the log canonical case [Kaw07]. A few years later, using the techniques developed in [BCHM10], Hacon extended Kawakita's result to higher codimensional log canonical centers [Hac14]. The purpose of this section is to give a rough sketch of Hacon's approach, limiting ourselves to the divisorial case. We will adopt the notation introduced in the previous section.

Assume by contradiction that $(S, \mathrm{Diff}(B))$ is log canonical, and $(X, S+B)$ is not log canonical along S. Using the minimal model program [KK10, Theorem 3.1], we can find a birational model $\pi : Y \to X$ satisfying the following:

- Y is $\mathbb{Q}$-factorial, i.e. every Weil divisor is $\mathbb{Q}$-Cartier;
- $\Delta_Y = \sum_i a_i D_i \geq 0$;
- $(Y, \Delta'_Y := \sum_{a_i \leq 1} a_i D_i + \sum_{a_i > 1} D_i)$, is dlt;
- every π-exceptional divisor appears in Δ_Y with coefficients at least 1.

We recall that a pair (Y, Δ'_Y) is called dlt if it is log canonical, and it is simple normal crossing along the locus where it is not plt [KM98, cf. Definition 2.37]. Such a model is called *dlt model* of (X, Δ). Note that Y is not smooth in general.

Let $\Gamma := \Delta'_Y - \tilde{S}$, and define $\Sigma := \Delta_Y - \tilde{S} - \Gamma$. As $(S, \mathrm{Diff}(B))$ is log canonical, $\Sigma \cap \tilde{S} = \emptyset$.

Now, fix an ample divisor H such that $K_Y + \tilde{S} + \Gamma + H$ is nef over X. Then, we can run a relative $(K_Y + \tilde{S} + \Gamma)$-MMP over X with scaling of H [BCHM10, cf. 3.10]. This produces a sequence of birational maps $\phi_i : Y_i \dashrightarrow Y_{i+1}$, which are flips or divisorial contractions, and induced morphisms $\pi_i : Y_i \to X$. For any divisor G on Y, we denote by G_i its strict transform on Y_i. In addition, there is a non-increasing sequence of rational numbers $\{s_i\}_{i \geq 0}$ such that:

- either $s_{N+1} = 0$ for some $N \in \mathbb{N}$, or $\lim_{i \to \infty} s_i = 0$;
- $K_{Y_i} + \tilde{S}_i + \Gamma_i + sH_i$ is π_i-nef for all $s_i \geq s \geq s_{i+1}$.

First, assume that $\tilde{S}_i \cap \Sigma_i \neq \emptyset$ for some $i \geq 0$. Then, we can write

$$(\pi_i^*(K_X + \Delta))|_{\tilde{S}_i} = K_{\tilde{S}_i} + \mathrm{Diff}(\Gamma_i + \Sigma_i).$$

As $\tilde{S}_i \cap \Sigma_i \neq \emptyset$, $(\tilde{S}_i, \mathrm{Diff}(\Gamma_i + \Sigma_i))$ is not log canonical. On the other hand, as $\tilde{S}_i$ is normal and

$$K_{\tilde{S}_i} + \mathrm{Diff}(\Gamma_i + \Sigma_i) = \pi_{\tilde{S}_i}^*(K_S + \mathrm{Diff}(B)),$$

$(\tilde{S}_i, \mathrm{Diff}(\Gamma_i + \Sigma_i))$ is log canonical. Thus, we get the required contradiction.

Therefore, we may assume $\tilde{S}_i \cap \Sigma_i = \emptyset$ for all $i \geq 0$. One can show that there exists $i_0 \in \mathbb{N}$ such that, for any $i \geq i_0$, $\tilde{S}_i \dashrightarrow \tilde{S}_{i+1}$ is an isomoprhism in codimension 1. Then, fix $m \gg 0$ such that $m\Sigma$ is an integral divisor. By the properties of the sequence $\{s_i\}_{i \geq 0}$, we can find $i \geq i_0$ such that $s_i > \frac{1}{m} \geq s_{i+1}$ and a divisor $\Theta_{i, \frac{1}{m}}$ such that $(Y_i, \Theta_{i, \frac{1}{m}})$ is klt and

$$H_i - m\Sigma_i - \tilde{S}_i \sim_{\mathbb{Q}, X} K_{Y_i} + \Theta_{i, \frac{1}{m}} + (m-1)\left(K_{Y_i} + \tilde{S}_i + \Gamma_i + \frac{1}{m} H_i\right).$$

As $K_{Y_i} + \tilde{S}_i + \Gamma_i + \frac{1}{m} H_i$ is nef over X, we can apply Kawamata-Viehweg vanishing [KM98, cf. proof of Corollary 2.68 and Theorem 2.70]. It follows that

$$R^1 \pi_{i,*} \mathcal{O}_{Y_i}(H_i - m\Sigma_i - \tilde{S}_i) = 0.$$

As a consequence, the morphism of sheaves

$$\pi_{i,*} \mathcal{O}_{Y_i}(H_i - m\Sigma_i) \to \pi_{\tilde{S}_i,*} \mathcal{O}_{S_i}(H_i - m\Sigma_i) = \pi_{\tilde{S}_i,*} \mathcal{O}_{S_i}(H_i)$$

is surjective.

On the other hand, for $m \gg 0$ the subsheaves

$$\pi_{i,*} \mathcal{O}_{Y_i}(H_i - m\Sigma_i) = \pi_{i_0,*} \mathcal{O}_{Y_{i_0}}(H_{i_0} - m\Sigma_{i_0}) \subset \pi_{i_0,*} \mathcal{O}_{Y_{i_0}}(H_{i_0})$$

are contained in $\mathcal{I}_{\pi_{i_0}(\Sigma_{i_0})} \cdot \pi_{i_0,*}\mathcal{O}_{Y_{i_0}}(H_{i_0})$. Since we are assuming that $(X, S+B)$ is not log canonical near S, we have $S \cap \pi_{i_0}(\Sigma_{i_0}) \neq \emptyset$. As $\tilde{S}_{i_0} \dashrightarrow \tilde{S}_i$ is an isomorphism in codimension 1, the induced homomorphism

$$\pi_{i,*}\mathcal{O}_{Y_i}(H_i - m\Sigma_i) \to \pi_{\tilde{S}_i,*}\mathcal{O}_{\tilde{S}_i}(H_i) = \pi_{\tilde{S}_{i_0},*}\mathcal{O}_{\tilde{S}_{i_0}}(H_{i_0})$$

is not surjective. Thus, we get the required contradiction.

This leads to a version of inversion of adjuction for log canonical pairs.

Theorem 5.1 (cf. Theorem 1.2)**.** *Let $(X, S+B)$ be a pair, where S is a normal prime divisor. Then, $(X, S+B)$ is log canonical near S if and only if $(S, \mathrm{Diff}(B))$ is log canonical.*

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