1. (Continuity part 1)

(a) For the graph indicated in the figure, determine whether the function is continuous at the points $a$, $b$, $c$ and $d$. Provide a brief explanation for each of these points (4 pts).
2. (Continuity part 2)

(a) Determine where the following functions are continuous

\[ \frac{3x-2}{8x-3} + \sin(x)e^x \] (1 pt)
\[ \tan(x)e^x \] (1 pt)
\[ f(g(x)), \text{ where } f(x) = \frac{1}{x} \text{ and } g(x) = e^{\sin(x)} \] (2 pts)

(b) Show that the following equations admit at least one solution in the interval given

(Hint: are the functions involved continuous? If so, what strategy can you choose in proving the claim?)

\[ x^5 - 27x + 3 = 12 \text{ for } x \in [0, 10] \] (2 pts)
\[ f(x) = \frac{1}{\pi} \text{ for } x \in [0, \frac{2}{\pi}], \text{ where} \]
\[ f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x \sin \left( \frac{1}{x} \right), & \text{if } x \neq 0 \end{cases} \]

(2 pts)

Solution:

(a) The function \( \sin(x)e^x \) is continuous, hence the points where \( \frac{3x-2}{8x-3} + \sin(x)e^x \) is not continuous coincide with the ones where \( \frac{3x-2}{8x-3} \) is not continuous. We are multiplying \( \tan(x) \) by \( e^x \), which is a continuous function and is never zero; hence the points where \( \tan(x)e^x \) is not continuous coincide with the ones where \( \tan(x) \) is not continuous.

The functions \( \sin(x) \) and \( e^x \) are continuous, and hence their composition \( e^{\sin(x)} \) too. The function \( \frac{1}{x} \) is continuous in \((0, +\infty)\). Since the range of \( e^{\sin(x)} \) is included in \((0, +\infty)\), the function is continuous on the whole real numbers.

(b) The function \( f(x) = x^5 - 27x + 3 \) is a polynomial, we check that \( f(0) \leq 12 \leq f(10) \) and we apply the intermediate value theorem.

The function \( f \) is continuous in \((-\infty, 0) \cup (0, +\infty)\) since there \( \frac{1}{x} \) is continuous, and hence \( \sin \left( \frac{1}{x} \right) \) and \( x \sin \left( \frac{1}{x} \right) \) too. By the squeezing theorem we can show it is continuous also in 0. Now we know we can apply the intermediate value theorem. We check \( f(0) \leq \frac{1}{\pi} \leq f \left( \frac{2}{\pi} \right) \) and conclude.
3. **(Evaluating derivatives)** Using the definition, determine the derivative function of the following functions.

(a) \(x^3 + 4x^2 - 2\) (1 pt)
(b) \(\sqrt{x}\) for \(x > 0\) (2 pts)
(c) \(\frac{1}{x-2}\) (2 pts)

**Solution:**

(a) We have

\[
(x + h)^3 + 4(x + h)^2 - 2 - x^3 - 4x^2 + 2 = x^3 + 3x^2h + 3xh^2 + h^3 + 4x^2 + 8xh + 8h^2 - 2 - x^3 - 4x^2 + 2 = 3x^2h + 3xh^2 + h^3 + 8xh + 8h^2
\]

Then dividing by \(h\) we have \(3x^2 + 3xh + h^2 + 8x + 8h\). Taking the limit as \(h\) goes to zero we have \(3x^2 + 8x\).

(b) We have

\[
\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}
\]

Taking the limit as \(h\) goes to zero we obtain \(\frac{1}{2\sqrt{x}}\).

(c) We have

\[
\frac{\frac{1}{x+h-2} - \frac{1}{x-2}}{h} = \frac{x - 2 - x - h + 2}{h(x+h-2)(x-2)} = \frac{-h(x+h-2)}{h} = -\frac{1}{(x+h-2)(x-2)}
\]

Taking the limit as \(h\) goes to zero we obtain \(-\frac{1}{(x-2)^2}\).
4. (Matching \( f \) and \( f' \)) Match the graph of the four functions given with the graph of their derivatives. Provide an explanation for each matching (8 pts).

Figure 1: function 1

Figure 2: function 2
Figure 3: function 3

Figure 4: function 4
Figure 5: derivative 1

Figure 6: derivative 2
Figure 7: derivative 3

Figure 8: derivative 4