1. Determine the domain of the following functions and show they are injective. Find their inverse \( f^{-1}(x) \), and indicate a valid domain and range of \( f^{-1}(x) \).

(a) \( f(x) = \frac{x}{x+1} \) (4 pts)
(b) \( f(x) = \ln(x^3 + 1) \) (4 pts)
(c) \( f(x) = \frac{1}{1+e^{-x}} \) (4 pts)

(a) The function is injective, since it passes the horizontal line test. The domain of \( f \) is \((-\infty, -1) \cup (-1, +\infty)\).

\[
f = \frac{x}{x+1} \quad (1) \\
(x+1)f = x \quad (2) \\
x f - x = -f \quad (3) \\
x (f - 1) = -f \quad (4) \\
x = f/(1-f) \quad (5)
\]

The inverse function is \( f^{-1}(x) = \frac{x}{1-x} \). The domain of \( f^{-1} \) is \((-\infty, 1) \cup (1, +\infty)\). The range is \((-\infty, -1) \cup (-1, +\infty)\).

(b) The function is injective since it is the composition of two injective functions (\( \ln(x) \) and \( x^3 + 1 \)). The domain of \( f \) is \((-1, +\infty)\).

\[
f = \ln(x^3 + 1) \quad (6) \\
ee^f = x^3 + 1 \quad (7) \\
x = (e^f - 1)^{1/3} \quad (8)
\]

The inverse function is \( f^{-1}(x) = \sqrt[3]{e^x - 1} \). The domain is all reals. The range is \((-1, +\infty)\).

(c) The function is injective since it is the composition of two injective functions (\( \frac{1}{1+x} \)
and $e^{-x}$). The domain of $f$ is all reals.

\[
f = \frac{1}{1 + e^{-x}} \tag{9}
\]

\[
(1 + e^{-x})f = 1 \tag{10}
\]

\[
f + e^{-x}f = 1 \tag{11}
\]

\[
e^{-x}f = 1 - f \tag{12}
\]

\[
e^{-x} = \frac{1 - f}{f} \tag{13}
\]

\[
e^x = \frac{f}{1 - f} \tag{14}
\]

\[
x = \ln \left( \frac{f}{1 - f} \right) \tag{15}
\]

The inverse function is $f^{-1}(x) = \ln \left( \frac{x}{1-x} \right)$. The domain of $f^{-1}$ is $(0, 1)$. The range is all reals.
2. Parametric functions: plot the parametric curve \((x(t), y(t))\) and use arrows to describe the direction of the movement as \(t\) increases

\[
x(t) = t^2 + t \quad y(t) = t^2 - t \quad -2 \leq t \leq 2
\]

(5 pts)
3. The horizontal motion of a pendulum in centimeters is indicated by the function \( x(t) = 10 \sin(2\pi t) \), where \( t \) is in seconds. Find the average velocity over the following time intervals

(a) \([1/4, 3/8]\) (1 pt)
(b) \([1/4, 5/16]\) (1 pt)
(c) \([1/4, 9/32]\) (1 pt)
(d) \([1/4, 17/64]\) (1 pt)
(e) \([1/4, 33/128]\) (1 pt)
(f) \([1/4, 65/256]\) (1 pt)

Based on your results, estimate the instantaneous velocity at \( t = 1/4 \) (2 pts). Hint: if you are not sure, try choosing a very narrow interval \([1/4, 1/4 + \epsilon]\), where \( \epsilon \) is very small.

(a)-(f) The formula one should use is the one of the difference quotient. If we denote by \( \bar{v} \) the average velocity we are interested in, we have

\[
\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1},
\]

where \( t_1 \) is the initial time and \( t_2 \) is the final time. In the exercise \( t_1 = \frac{1}{4} \) for all points (a)-(f), while \( t_2 \) varies \((\frac{3}{8}, \frac{5}{16}, \frac{9}{32}, \frac{17}{64}, \frac{33}{128}, \frac{65}{256})\). The function \( x(t) \) is \( x(t) = 10 \sin(2\pi t) \). The values of the average velocity are -23.4315, -12.1793, -6.1487, -3.0818, -1.5418, -0.7710. The instantaneous velocity is zero. The approximations approach that slowly.