PERIODIC STRUCTURES IN A VAN DER WAALS FLUID

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Abstract

A system of partial differential equations modeling a van der Waals fluid or an elastic medium with nonmonotone pressure-density relation is studied. As the system changes type, regularizations are considered. The existence of one-dimensional periodic travelling waves with prescribed average density in a certain range, average velocity, and wavelength is proved. They exhibit layer structure when the regularization parameter is small. Similarities with the Cahn-Hilliard equation are explored.

1 Introduction

We consider a one-dimensional isothermal elastic medium with density $\rho(x,t)$, velocity $u(x,t)$, and a nonmonotone pressure-density relation $p = p(\rho)$. It obeys a system of evolution equations of the form

$$\rho_t + (\rho u)_x = 0, \tag{1}$$

$$u_t + uu_x + \frac{p'(\rho)}{\rho}\rho_x = 0, \tag{2}$$

which is hyperbolic or elliptic according as $p'(\rho) > 0$ or $p'(\rho) < 0$. Our concern is with the case when the system changes type, being hyperbolic for small or large ρ and elliptic in between. The two regions of hyperbolicity are usually associated with two different phases. For example in a van der Waals fluid, which is described this way with a specific choice of function p, the left ascending branch of this function represents the gaseous state, and the right hand one represents liquid.

Numerical studies by Hsieh and Wang [7] have shown that a regularized version of this system, in which second derivatives with small parameters appear on the right sides, exhibit solutions with multiple thin layers representing phase transitions, and that some such configurations move with constant velocity. Those numerical solutions motivated us to consider, analytically, traveling waves of regularized versions of (1), (2).

We consider two kinds of regularization. In one case, second order space derivatives with small coefficients are adjoined on the right of both equations:
R1:

$$\rho_t + (\rho u)_x = \epsilon\rho_{xx}, \tag{3}$$

$$u_t + uu_x + \frac{p'(\rho)}{\rho}\rho_x = \epsilon u_{xx}. \tag{4}$$

The other is a regularization of viscosity-capillarity type stemming from an idea of Korteweg [8]:
R2:

$$\rho_t + (\rho u)_x = 0, \tag{5}$$

$$u_t + uu_x + \frac{p'(\rho)}{\rho}\rho_x = \epsilon(u + \epsilon A \rho_x)_{xx}, \quad A > 0. \tag{6}$$

A term with the same appearance as that in (6) was proposed and studied in the context of the Lagrangian version of (1), (2) by Serrin [11], Slemrod [12] and Hagan and Slemrod [6]. It is examined here in the Eulerian form. The terms in u_{xx} and ρ_{xxx} are suggestive of viscous and capillary contributions to the stress, respectively.

Our results are based in part on qualitative properties of the resulting traveling wave equations; it will be clear that they are valid for regularizations more general than the specific ones written here, e.g. nonlinear ones.

It turns out that in each case, the study of periodic traveling waves bears a great similarity to that of 1D stationary solutions of the Cahn-Hilliard equation
CH:

$$\rho_t = \left[-\epsilon^2 \rho_{xx} + q(\rho)\right]_{xx}, \tag{7}$$

for a bistable function q. The variable ρ has typical meaning as a solute concentration in an alloy rather than mass density in that context. This equation has been the subject of a large number of investigations. Relevant to this paper are the results of Carr, Gurtin, and Slemrod [2], who gave a very careful and detailed existence and local uniqueness proof of periodic layered solutions for small ϵ with prescribed average value of ρ and prescribed wavelength. Their results included exponential estimates for various properties of the solutions. The proof was based on an implicit function theorem argument applied to an equation obtained through carefully chosen transformations. An exact count of the number of stationary solutions for all values of ϵ was obtained for a specific choice of $q(\rho) = -\rho + \rho^3$ in the papers of Grinfeld and Novick-Cohen [5] and Novick-Cohen and Peletier [10]; see also the references therein. However their argument cannot be easily extended to more general functions $q(\rho)$.

Also relevant is the paper by Grant [3], who studied the role of periodic solutions (not necessarily layered) in the unfolding of the instabilities of constant solutions, a process analogous to spinodal decomposition in materials scientific contexts. There are recent important extensions by Maier and Wanner [9], which include problems in two space dimensions.

We are concerned, first of all, with the existence question for traveling waves with prescribed average values of ρ and u for systems R1 and R2, and for stationary solutions of CH, with general function p of the type described above (see also (8) below). In the case of the viscosity-capillarity regularization (5)-(6), the problem can be reduced to a system of ODE's with constraint which is formally identical to the corresponding system in the Cahn-Hilliard case. The other regularization (3)-(4) yields a system which, while not identical, nevertheless has important qualitative properties in common with the equations coming from the Cahn-Hilliard model. The existence of layered periodic traveling waves for small enough ϵ with prescribed average value $\bar{\rho}$ of ρ in a certain range and average value $\bar{u}$ of u can be given by an easy adaptation of the proof in [2]. However, we provide here in secs. 2 and 3 an alternative proof for a much larger range of ϵ which is much simpler, though it lacks the uniqueness and exponential estimates which one gets with the implicit function theorem argument. Our proof does not use that theorem, but rather a continuity argument. When $p'(\rho)$ is unimodal, the allowed ranges of ϵ and $\bar{\rho}$ appear to be the largest possible. We merely require ϵ to be smaller than an explicit number depending on the maximum value of $-p'(\rho)$, and $\bar{\rho}$ to lie in an explicit interval depending on ϵ. Alternatively, if $\bar{\rho}$ lies in another explicit interval, then such a periodic solution exists for sufficiently small ϵ.

Also in sec. 3, we characterize the limit, as $\epsilon \to 0$, of our periodic solutions as specific piecewise constant solutions of (1), (2), thus providing a selection criterion for such solutions on the basic of vanishing regularization.

The formal similarity among the three models (3)-(4), (5)-(6), and (7) seen when dealing with the existence problem for steady periodic solutions breaks down in large part when we pass to dynamical considerations. In fact the dynamics in the three cases are considerably different from one another. For one thing, the van der Waals models lack the gradient flow nature of (7). However, it turns out that the three dispersion relations for unstable constant solutions are qualitatively similar.

The dispersion relation gives the growth (or decay) rate σ of solutions of the equation linearized about the given unstable constant, as a function of the wave number k of the mode. In all cases, σ attains a positive maximum at a nonzero value of $k = O(\epsilon^{-1})$. It has long been known, for the Cahn-Hilliard equation, that this fact is fundamental to a process by which patterned solutions arise from unpatterned initial states. A rigorous theory of this effect was given in [3]. It will be argued in sec. 4 that the regularized van del Waals models no doubt exhibit the same pattern-forming capability.

2 Traveling waves

First, we consider the problem of traveling waves for R1, in which the average density $\overline{\rho}$ and fluid velocity $\overline{u}$ are prescribed quantities. In the case that ϵ is small, these solutions exhibit layers where the density jumps between an upper (liquid) and a lower (gaseous) state. There is no loss of generality in taking $\overline{u} = 0$, for the system (3), (4) is invariant under the transformation $u \to u - \overline{u}, \;\; x \to x - \overline{u}t$. This transformation is to a moving coordinate frame whose velocity is $\overline{u}$.

We assume that $p(\rho)$ is a C^2 function defined for positive ρ such that for some positive pair $\rho_1 < \rho_2$,

$$p'(\rho) > 0 \;\text{ for }\; \rho \in (0, \rho_1) \cup (\rho_2, \infty) \;\text{ and }\; p'(\rho) < 0 \;\text{ for }\; \rho \in (\rho_1, \rho_2). \tag{8}$$

The indefinite integral

$$q(\rho) = \int \rho^{-1} p'(\rho) d\rho$$

is of course strictly increasing in the former set and strictly decreasing in the latter. Thus $q(\rho)$ plus a suitable constant can be considered a generic bistable function. The indefinite integral, in fact, is defined only up to an additive constant, but we now specify that constant so that q has exactly three zeros, $0 < \rho_- < \rho_0 < \rho_+$, with

$$\int_{\rho_-}^{\rho_+} q(\rho) d\rho = 0. \tag{9}$$

For definiteness, the wave length of the traveling waves we construct will be taken to be unity. This involves no loss of generality, since the wavelength can be adjusted by changing ϵ. The profiles will have only one maximum point for ρ and one minimum point in each wavelength. We let c be the unknown speed and write the traveling wave equations

$$-c\rho' + (\rho u)' = \epsilon \rho'', \tag{10}$$

$$-cu' + (\frac{1}{2}u^2 + q(\rho))' = \epsilon u'', \tag{11}$$

primes denoting differentiation with respect to the traveling wave coordinate $z = x - ct$. This system is equivalent to the following integrated system, with arbitrary integration constants $\bar{a}$ and $\bar{b}$:

$$\epsilon \rho' = \rho(u - c) + \bar{a}, \tag{12}$$

$$\epsilon u' = q(\rho) + \frac{1}{2}u^2 - cu + \bar{b}. \tag{13}$$

Our solutions will have $\bar{a} = 0$, because only in this case can one obtain a heteroclinic pair from a heteroclinic orbit in the simple way described below. Physically, this condition means that the average mass flux relative to the traveling wave frame is 0. This choice of $\bar{a}$ is relevant to the discovery by Grinfeld [4] of traveling fronts with cavitation (see sec. 5).

We change variables, setting $\zeta = z/\epsilon$ and

$$w = (u - c)\rho, \tag{14}$$

to obtain (dots mean differentiation with respect to ζ)

$$\dot{\rho} = w, \tag{15}$$

$$\dot{w} = \rho(q(\rho) + b) + \frac{3}{2\rho}w^2, \tag{16}$$

where

$$b = \bar{b} - \frac{1}{2}c^2. \tag{17}$$

We are interested in solutions periodic in ζ with minimal period (wavelength) $1/\epsilon$ and with given $\bar{\rho}$ and $\bar{u} = 0$. We first construct periodic solutions of (15), (16) with that period such that ρ has the required average value. In this process, the constant b is chosen appropriately. Then we show that $\bar{b}$ and c can be chosen to produce, on the basis of (14), a periodic function u with the specified average value such that (ρ, u) satisfies (10), (11).

In a similar way, we may write the traveling wave equations for R2. Again setting $\bar{a} = 0$, we find $u = c$, and the following system replaces (15), (16):

$$\dot{\rho} = w, \tag{18}$$

$$\dot{w} = \frac{1}{A}q(\rho) + b, \tag{19}$$

where now $w = \dot{\rho}$ and $b = -\frac{1}{A}\left(\frac{1}{2}c^2 + \bar{b}\right)$.

Finally, stationary solutions of CH satisfy

$$\dot{\rho} = w,$$

$$\dot{w} = q(\rho) + b.$$

These are all special cases of the system

$$\epsilon\dot{\rho} = w, \tag{20}$$

$$\epsilon\dot{w} = \alpha(\rho)(q(\rho) + b) + \beta(\rho)w^2, \tag{21}$$

where $\alpha > 0$ and β are given functions of ρ. This general formulation with arbitrary α and β can be treated as we do the special version in this section; and the results are essentially the same. We provide details for only the case of R1.

Note that the critical points of the system (15), (16) are the points $(\rho_i(b), 0)$ with $i = -, 0, +$, where $\rho_-(b) < \rho_0(b) < \rho_+(b)$ are the three roots of

$$q(\rho) + b = 0. \tag{22}$$

By the definition of q, they exist for $b = 0$; and by the bistable assumption on q, they exist also for b in some open interval B containing 0. We take B to be the maximal such interval.

For $b \in B$, it is easily checked by a linearization analysis (or alternatively from the analysis below) that the two critical points $(\rho_\pm(b), 0)$ of the system are saddle points, the one-dimensional unstable manifold $M(b)$ of the point $(\rho_-(b), 0)$ intersecting the ρ-axis in the (ρ, w) plane with a finite positive slope at that point. It therefore continues at least a short distance in the (ρ, w) phase plane above and to the right of the left hand saddle point.

Also note that the system is invariant under the transformation $\zeta \to -\zeta,\ w \to -w,\ \rho \to \rho$. It follows that if there exists a heteroclinic orbit $(\rho^f(\zeta), w^f(\zeta))$ connecting $(\rho_-(b), 0)$ to $(\rho_+(b), 0)$, then there also exists a reverse connection, namely $\rho^r(\zeta) = \rho^f(-\zeta),\ \ w^r(\zeta) = -w^f(-\zeta)$. Such a pair of connections is called a heteroclinic pair.

We define the functions

$$\phi(\rho, b) = 2\int_{\rho_-(b)}^{\rho} s^{-2}(q(s) + b)ds, \tag{23}$$

$$\Psi(\rho, w; b) = \frac{w^2}{\rho^3} - \phi(\rho, b). \tag{24}$$

For each $b \in B$, ϕ is a double-well function of ρ with minima at $\rho_\pm(b)$ and maximum at $\rho_0(b)$.

Lemma 1 **(a)** *For each $b \in B$, the trajectories of (15), (16) are the level curves of the function $\Psi(\rho, w; b)$, in view of the double well nature of the function ϕ..*

(b) *There exists exactly one value $b = b_0$ such that*

$$\Psi(\rho_+(b_0), 0; b_0) = 0.$$

Proof: It can be verified directly that (15), (16) has

$$\Psi(\rho, w; b) + k = 0 \tag{25}$$

as a first integral, where k is an arbitrary constant. This establishes (a). Claim (b) follows from the fact that $\phi(\rho, b)$ is a strictly increasing function of b (note that the lower limit $\rho_-(b)$ is a decreasing function), plus the facts that $\phi(\rho_+(b), b) < 0$ for b near the left endpoint of B, and the opposite inequality holds near the right endpoint. □

In the case of the general model (20), (21), the definitions (23), (24) are to be replaced by

$$\phi(\rho, b) = 2\int \alpha(\rho)\exp\left(-2\int \beta(\rho)d\rho\right)(q(\rho) + b)d\rho,$$

$$\Psi(\rho, w; b) = w^2 \exp\left(-2\int \beta(\rho)d\rho\right) - \phi(\rho, b).$$

3 Periodic orbits

For $b \in B$, we shall also use the functions

$$k^*(b) = \max_{\rho \in [\rho_-(b), \rho_+(b)]} \phi(\rho, b) = \phi(\rho_0(b), b)$$

$$k^0(b) = \begin{cases} \phi(\rho_+(b), b) & \text{for} \quad b > b_0 \\ 0 & \text{for} \quad b \le b_0 \end{cases}$$

Theorem 1 **(a)** *For $b = b_0$, (25) defines a heteroclinic pair for $k = 0$, a family of periodic orbits for $0 < k < k^*(b)$ and one point for $k = k^*(b)$.*

(b) *If $b < b_0$, (25) defines a homoclinic orbit for $k = 0$, a family of periodic orbits for $0 < k < k^*(b)$ and one point for $k = k^*(b)$.*

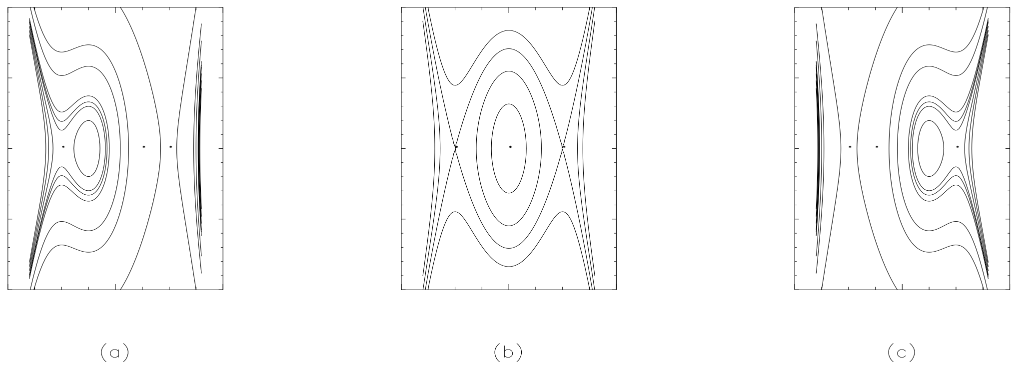

Figure 1: Phase portraits in the (ρ, w) plane. (a) $b < b_0$; (b) $b = b_0$; (c) $b > b_0$.

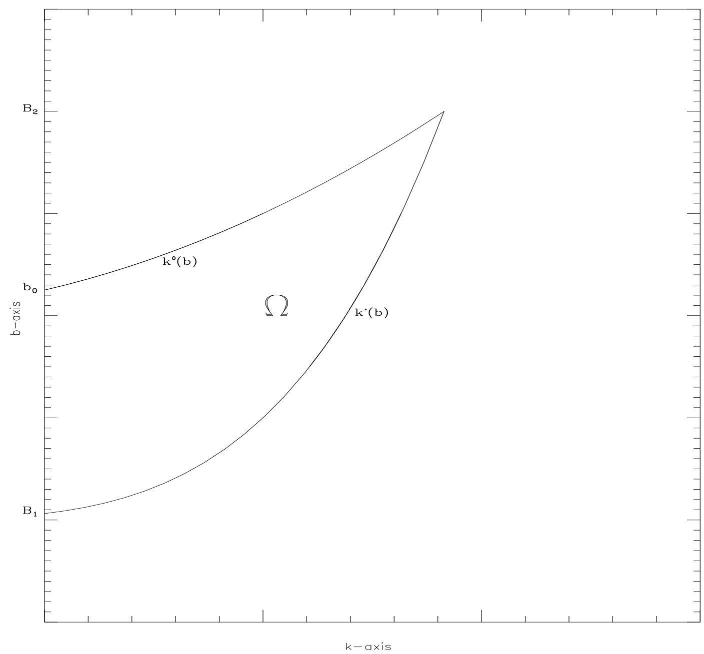

Figure 2: The set Ω

(c) *If $b > b_0$, (25) defines a homoclinic orbit for $k = k^0(b)$, a family of periodic orbits for $k^0(b) < k < k^*(b)$ and one point for $k = k^*(b)$.*

Proof. These assertions are obvious from the geometry of the level curves of Ψ, in view of the double well nature of ϕ.

The phase portraits are shown in Fig. 1. They are qualitatively the same as for the very well studied simpler system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= (x^2 - 1)(x + b). \end{aligned}$$

The above theorem can be summarized graphically by Fig. 2. Consider the closed region Ω in the (k, b) plane bounded by the curves $k = k^*(b)$ and $k = k^0(b)$. These two curves merge at the two endpoints $b = B_1,\ b = B_2$ of $B = (B_1, B_2)$. In fact, the upper endpoint of B is characterized by

$$\lim_{b\uparrow B_2} \rho_+(b) = \lim_{b\uparrow B_2} \rho_0(b) = \rho_2. \tag{26}$$

Similarly,

$$\lim_{b\downarrow B_1} \rho_-(b) = \lim_{b\downarrow B_1} \rho_0(b) = \rho_1. \tag{27}$$

For each (k, b) in Ω_0 (the interior of Ω), (25) defines a periodic orbit by Thm. 1.

Now from (15), (24) and (25), we have

$$\frac{d\zeta}{d\rho} = (\rho^3(\phi(\rho, b) - k))^{-1/2} \equiv h(\rho, k, b). \tag{28}$$

Therefore the periods $\lambda(k, b)$ and the average values $R(k, b)$ of ρ of the periodic orbits are given by the following well known expressions:

$$\lambda(k, b) = 2\int_{\rho_-^*}^{\rho_+^*} h(\rho, k, b)d\rho, \tag{29}$$

$$\lambda(k, b)R(k, b) = \int_{\rho_-^*}^{\rho_+^*} \rho h(\rho, k, b)d\rho \tag{30}$$

where $\rho_\pm^*(k, b)$ satisfy $\Psi(\rho_\pm^*, 0; b) + k = 0$, i.e. $\phi(\rho_\pm^*, b) = k$. It is easy to see that when $(k, b) \in \Omega_0$, $\lambda(k, b)$ and $R(k, b)$ are finite since $\rho_\pm^*$ are simple zeros of $\phi - k$ and the integrals converge.

Lemma 2 *$\lambda(k, b)$ and $R(k, b)$ are continuously differentiable functions for (k, b) in the interior of Ω. Moreover,*

(a) *For $b < b_0$ fixed, $(R(k, b), \lambda(k, b)) \to (\rho_-(b), +\infty)$ as $k \to 0$.*

(b) *For $b > b_0$ fixed, $(R(k, b), \lambda(k, b)) \to (\rho_+(b), +\infty)$ as $k \to k^0(b)$.*

Remark: Because of this, we can and shall take the functions λ and R to be defined also on the right boundary $\partial_r\Omega$ of Ω.

Proof: First, we show the differentiability of λ. Let (k, b) be in the interior of Ω, so that $\phi_\rho(\rho_\pm^*(k, b), b) \neq 0$ and $\frac{d\rho_-^*}{dk} < \infty$. For some constant δ so small that $\phi_\rho(\rho, b) \neq 0$ for $\rho \in [\rho_-^*(k, b), \rho_-^*(k, b) + \delta] \cup [\rho_+^*(k, b) - \delta, \rho_+^*(k, b)]$, we write

$$\lambda(k, b) = 2\int_{\rho_-^*}^{\rho_-^*+\delta} h(\rho, k, b)d\rho + 2\int_{\rho_-^*+\delta}^{\rho_+^*-\delta} h(\rho, k, b)d\rho + 2\int_{\rho_+^*-\delta}^{\rho_+^*} h(\rho, k, b)d\rho = \lambda_1(k, b) + \lambda_2(k, b) + \lambda_3(k, b).$$

Defining $y = \rho - \rho_-^*$, we have

$$\lambda_1(k, b) = 2\int_0^\delta h(\rho_-^*(k, b) + y, k, b)dy. \tag{31}$$

We fix b and define the function

$$m(\rho, y) = (\rho + y)^3(\phi(\rho + y) - \phi(\rho)) > K_1 y \tag{32}$$

for y small when $\phi'(\rho) > 0$. Here K_1 (and K_i below) are positive constants independent of y for $0 \leq y \leq \delta$. By (28), the integrand in (31) is just $[m(\rho_-^*, y)]^{-1/2}$. We also have

$$\left|\frac{\partial m}{\partial \rho}\right| = \left|3(\rho + y)^2(\phi(\rho + y) - \phi(\rho)) + (\rho + y)^3(\phi'(\rho + y) - \phi'(\rho))\right| < K_2|y|, \tag{33}$$

so that from (32) and (33),

$$\left|\frac{\partial}{\partial k}h(\rho_-^*(k)+y,k)\right| = \left|\frac{1}{2}m^{-3/2}\frac{\partial m}{\partial \rho}(\rho_-^*)\frac{d\rho_-^*}{dk}\right| \leq K_3 y^{-1/2}.$$

It follows from this and (31) that λ_1 is differentiable with respect to k, with $\left|\frac{\partial \lambda_1}{\partial k}\right| \leq K_4 \delta^{1/2}$.

The same argument establishes the differentiability of λ_1 with respect to b, and that of λ_3 with respect to k and b.

Finally, the integrand in the definition of $\lambda_2(k)$ is bounded away from any singularity, and therefore $\lambda_2(k,b)$ is smooth.

The differentiability of the integral on the right of (30), hence of $R(k,b)$, is also given by the same argument.

We now prove part (a) of the lemma. For $b < b_0$, we have $\rho_-^*(k,b) \to \rho_-(b)$ and $\rho_+^*(k,b) \to \tilde{\rho}(b)$ as $k \to 0$ where $\tilde{\rho}(b)$ satisfies $\phi(\tilde{\rho},b) = 0$. Therefore

$$\lambda(k,b) > \int_{\rho_-^*}^{\rho_+^*} h(\rho,k,b)d\rho \to \int_{\rho_-}^{\tilde{\rho}} \frac{1}{\sqrt{\rho^3\phi}}d\rho = \infty,$$

since ρ_- is a double zero of ϕ and $\frac{1}{\sqrt{\phi}} \approx \frac{1}{(\rho-\rho_-)}$ as $\rho \to \rho_-$. To prove the limit for $R(k,b)$, take any $\delta > 0$. We have

$$R = \frac{\int_{\rho_-^*}^{\rho_-+\delta} \rho h(\rho,k,b)d\rho + \int_{\rho_-+\delta}^{\rho_+^*} \rho h(\rho,k,b)d\rho}{\lambda(k,b)} = I_1 + I_2.$$

For k sufficiently small, we have that $h(\rho,k,b)$ is bounded independently of k for $\rho > \rho_- + \delta$, hence the numerator is as well, and for some N,

$$I_2 \leq \frac{N}{\lambda} \to 0 \text{ as } k \to 0.$$

For I_1, we have

$$\frac{\rho_-^* \int_{\rho_-^*}^{\rho_-+\delta} h(\rho,k,b)d\rho}{\lambda} \leq I_1 \leq \frac{(\rho_-+\delta)\int_{\rho_-^*}^{\rho_-+\delta} h(\rho,k,b)d\rho}{\lambda}$$

and using an argument similar to the one used for I_2, we have

$$\frac{\int_{\rho_-^*}^{\rho_-+\delta} h(\rho,k,b)d\rho}{\lambda} = 1 - \frac{\int_{\rho_-+\delta}^{\rho_+^*} h(\rho,k,b)d\rho}{\lambda} \to 1.$$

Therefore

$$\rho_- \leq \liminf_{k\to\infty} I_1 \leq \limsup_{k\to\infty} I_1 \leq \rho_- + \delta$$

Letting $\delta \to 0$, we have that $R \to \rho_-$, hence part (a).

The proof of (b) is similar. □

Referring to the points ρ_1 and ρ_2 in (8), we assume below that for some point $\rho_3 \in (\rho_1,\rho_2)$,

$$p'(\rho) \text{ is a strictly decreasing function for } \rho \in (\rho_1,\rho_3) \text{ and is strictly increasing for } \rho \in (\rho_3,\rho_2). \tag{34}$$

If (34) does not hold, similar but weaker results hold by the same method of proof.

Lemma 3 *On the right hand boundary $\partial_r\Omega$ of Ω, we have*

$$\lambda(k^*(b),b) = \frac{2\pi}{\sqrt{-p'(\rho_0(b))}}, \tag{35}$$

$$R(k^*(b),b) = \rho_0(b). \tag{36}$$

Moreover, $\rho_0(b)$ is a strictly increasing function of b, and if (34) holds, $\lambda(k^*(b),b)$ decreases strictly to a minimal value λ_m (given by (35) with $\rho_0(b) = \rho_3$), then increases to ∞, as b ranges from B_1 to B_2. Hence for each $\lambda > \lambda_m$ it attains the value λ at exactly two values of b.

Proof. As $k \to k^*(b)$, the orbits of the system (15)-(16) approach the critical point $(\rho_0(b), 0)$, which is a center. To obtain the wavelength of orbits near such a center is a well known calculation (see e.g. [5]), which yields (35). The value of ρ at this critical point becomes the average value of ρ on the orbit; hence (36). As b ranges from B_1 to B_2, $\rho_0(b)$ increases monotonically from ρ_1 to ρ_2. This fact, together with (34) and (35), establish the unimodal nature of $\lambda(k^*(b), b)$. $\square$

For $\lambda > \lambda_m$, let $\tilde{\rho}_1(\lambda) < \tilde{\rho}_2(\lambda)$ be the two values of ρ at which $\frac{2\pi}{\sqrt{-p'(\rho)}} = \lambda$.

Theorem 2 *Let $\lambda^* > \lambda_m$ and $R^* \in (\tilde{\rho}_1(\lambda^*), \tilde{\rho}_2(\lambda^*))$. There exists a number $b = b^*$ such that (15), (16) has a periodic solution with wavelength λ^* and $\bar{\rho} = R^*$.*

Proof. We denote $p = (k, b)$. Let $\Gamma = \{p \in \Omega : \lambda(p) = \lambda^*\}$ (closed). By Lemma 3, Γ contains exactly two points $p_1,\ p_2 \in \partial_r\Omega$. Let p_1 be the "lower" one (with the lesser value of b). For $\delta > 0,\ 0 < \alpha < 1$, we let S be the union of (a) the open disks D_1 and D_2 centered at p_1 and p_2 of radii δ, and (b) a finite open covering of Γ by disks of radius $\alpha\delta$ centered at points of Γ. For fixed δ, α can and will be chosen so small that $(S \setminus (D_1 \cup D_2)) \cap \partial_r\Omega = \emptyset$; this is because $\Gamma \setminus (D_1 \cup D_2)$ lies a positive distance from $\partial_r\Omega$. Let Γ' be the connected component of ∂S containing some point on ∂D_2 not in Ω. We traverse Γ' counterclockwise from that point and order the points on Γ' accordingly. There will be points q_i (at least one of them) where Ω_0 is entered, and q_i' where Γ' leaves Ω_0, with $q_i < q_i' \le q_{i+1}$. All points q_i and q_i' lie on $\partial D_1 \cap \Gamma'$ or $\partial D_2 \cap \Gamma'$, by the above choice of α. Since $\Gamma' \cap \Omega$ is bounded away from Γ, the function $\lambda(p) - \lambda^*$, defined for $p \in \Gamma' \cap \Omega$, cannot vanish. But it may change sign between q_i', where Γ' leaves Ω, and q_{i+1}, which is the next reentry point (it is undefined between these two points). In fact it must change sign in this way for some i, because there are points on $\partial D_2 \cap \Gamma'$ ($\in \partial_r\Omega$) where $\lambda(p) - \lambda^*$ has both signs.

Now q_i' and q_{i+1} both lie on ∂D_1 or both on ∂D_2, because all points on Γ' not in Ω lie on one or the other. If they both lie on ∂D_2, there can be no sign change; in fact the arc of ∂D_2 from q_i' to q_{i+1} is traversed counterclockwise, and if there were a sign change, it would start above p_2 and end below. This would contradict the fact that the segment extending left from p_2 lies in Ω. Therefore for some $i = i_0$, $q_{i_0}' \in \partial D_1$. Take it to be the first such i. Thus the part of Γ' between q_{i_0} and q_{i_0}' lies in Ω and connects D_1 to D_2. Choose δ so small that $R^* \in (R(q_{i_0}'), R(q_{i_0}))$. Thus there exists a point $p_\delta^* \in \Gamma' \cap \Omega$ at which $R(p_\delta^*) = R^*$. The conclusion is obtained by letting $\delta \to 0$ and taking a subsequence of the p_δ^*. They converge to a point p^* where $R(p^*) = R^*$ and $\lambda(p^*) = \lambda^*$. $\square$

Theorem 3 *Let $\epsilon < \lambda_m^{-1}$, where λ_m was given in Lemma 3, let $\lambda = \epsilon^{-1}$ and finally let $\bar{\rho} \in (\tilde{\rho}_1(\lambda), \tilde{\rho}_2(\lambda))$. There exists a periodic traveling wave solution $(\rho(z;\epsilon),\ u(z;\epsilon),\ c(\epsilon))$ of (10), (11) with minimal period 1, average value of ρ equal to $\bar{\rho}$, and average value of u equal to 0. Moreover,* $\lim_{\epsilon\downarrow 0} c(\epsilon) = 0$,

$$\lim_{\epsilon\downarrow 0} \rho(z;\epsilon) = \begin{cases} \rho_-(b_0) \text{ for } 0 < z < \alpha, \\ \rho_+(b_0) \text{ for } \alpha < z < 1, \end{cases} \tag{37}$$

and

$$\lim_{\epsilon\downarrow 0} u(z;\epsilon) = 0, \quad z \notin \{0, \alpha, 1\}, \tag{38}$$

where $\alpha = \frac{\rho_+(b_0) - \bar{\rho}}{\rho_+(b_0) - \rho_-(b_0)}$.

Proof. Consider the periodic function $(\rho^\epsilon(\zeta),\ w^\epsilon(\zeta))$ found in Theorem 2 for $R^* = \bar{\rho}$ and $\lambda^* = \epsilon^{-1}$. If we integrate (15) over one period, we get that the average $\overline{w^\epsilon} = 0$. In view of (14), we know that in order for $\overline{u^\epsilon} = 0$, it suffices that

$$0 = c^\epsilon + \overline{w^\epsilon / \rho^\epsilon}. \tag{39}$$

This equation determines c^ϵ uniquely, hence $u^\epsilon(\zeta)$ (14) as well as the parameter $\bar{b}$ from (17). The pair $(\rho^\epsilon(\zeta), u^\epsilon(\zeta))$ so obtained becomes a solution $(\rho^\epsilon(z), u^\epsilon(z))$ of (10), (11) with period 1 when we set $\zeta = z/\epsilon$. This concludes the proof of existence.

We now let $\epsilon \to 0$, both the averages remaining fixed.

For any small enough $\epsilon, \delta > 0$, let $\ell_\delta^\epsilon = \{z \mid \rho_-(b_0) + \delta < \rho^\epsilon(z) < \rho_+(b_0) - \delta\}$. The length of ℓ_δ^ϵ is given, via (28), by

$$|\ell_\delta^\epsilon| = \int_{\ell_\delta^\epsilon} dz = \epsilon \int_{\rho_-(b_0)+\delta}^{\rho_+(b_0)-\delta} h(\rho, b_\epsilon^*, k_\epsilon^*) d\rho = \epsilon I_\epsilon.$$

Since $k_\epsilon^* \to 0$ and $b_\epsilon^* \to b_0$ as $\epsilon \to 0$, we have

$$I_\epsilon \to \int_{\rho_-(b_0)+\delta}^{\rho_+(b_0)-\delta} \frac{1}{\sqrt{\rho^3 \phi(\rho, b_0)}} d\rho < \infty.$$

Therefore $|\ell_\epsilon^\delta| \to 0$ as $\epsilon \to 0$.

In the following the symbol $\omega(\epsilon, \delta)$ will denote several different positive functions of those two variables with the property

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \omega(\epsilon, \delta) = 0$$

For example, we have found that $|\ell_\epsilon^\delta| = \omega(\epsilon, \delta)$.

We shift the independent variable z so that for all ϵ, $\rho^\epsilon(0) = \rho_0(b_0)$, $w^\epsilon(0) < 0$, where ρ_0 was defined at the beginning of sec. 3. Let $L_-(\epsilon, \delta)$ be the interval where $\rho^\epsilon(z) < \rho_-(b_0) + \delta$, and $L_+(\epsilon, \delta)$ be the same where $\rho^\epsilon(z) > \rho_+(b_0) - \delta$.

It follows that

$$R_0 = \int_0^1 \rho^\epsilon(z) dz = \int_{L_-} [\rho_-(b_0) + (\rho^\epsilon(z) - \rho_-(b_0))] dz + \int_{L_+} [\rho_+(b_0) + (\rho^\epsilon(z) - \rho_+(b_0))] dz + \int_\ell \rho^\epsilon(z) dz$$

$$= \rho_-(b_0)|L_-| + \rho_+(b_0)|L_+| + \omega(\epsilon, \delta)$$

$$= \rho_-(b_0)|L_-| + \rho_+(b_0)(1 - |L_-|) + \omega(\epsilon, \delta),$$

since $L_- + L_+ + \ell = 1$.

Taking the limit as ϵ, then $\delta \to 0$, we find in the limit that $|L_-| = \alpha$, as defined above. This proves (37).

Since we have

$$\frac{w^2}{\rho^3} - 2 \int_{\rho_-(b^*)}^{\rho} s^{-2}(q(s) + b_\epsilon^*) ds + k_\epsilon^* = 0$$

and $k_\epsilon^* \to 0$, $b_\epsilon^* \to b_0$ as $\epsilon \to 0$, (37) and the characterization of b_0 given in Lemma 1 imply $w(z; \delta) \to 0$ as $\epsilon \to 0$ for $z \notin \{0, \alpha, 1\}$. We now obtain from (39) that $c_\epsilon \to 0$, hence on the basis of (14) that (38) holds. □ The following is immediate.

Corollary Let $\bar{\rho} \in (\rho_-, \rho_+)$. Then for small enough ϵ, there is a traveling wave solution of (10), (11) with this value of $\bar{\rho}$, with $\bar{u} = 0$, and with minimal period 1.

4 Evolution to a structured state and comparison with Cahn-Hilliard theory

We have seen a strong formal similarity between the mathematics of stationary traveling waves of the van del Waals fluid model, regularized in either of two ways, and stationary solutions of the Cahn-Hilliard equation.

When we look at the dynamics of the three models, however, we see a considerable difference. None of the evolution systems (3)-(4), (5)-(6), or (7) can be reduced to another. Despite this, we shall see that there is strong evidence that in each case, structured solutions such as we have discovered here can arise from a "spinodal decomposition" process—the unfolding of the instability of a constant solution.

In fact, in each case, the constant $\rho \equiv \rho_0, \ \ u \equiv 0$ is an exact solution, and it is unstable provided that $q'(\rho_0) < 0$. We may look for exponential solutions

$$\rho(x,t) = \rho_0 + \delta R e^{ikx+\sigma t}, \quad u(x,t) = \delta U e^{ikx+\sigma t} \tag{40}$$

to the linearizations of these systems about the unstable constant. They exist when σ assumes one of two possible values depending on k. The value with the larger real part provides the dispersion relation for the constant solution. This dispersion relation $\sigma(k)$ is readily calculated in each of the three cases. We find

For (3)-(4):

$$\sigma(k) = ak - \epsilon k^2, \tag{41}$$

where we have set $a = \sqrt{-\rho_0 q'(\rho_0)}$.

For (5)-(6):

$$\sigma(k) = \frac{1}{2} k \left\{ \sqrt{4a^2 + \epsilon^2 k^2 - 2A\epsilon^2 k^4 \rho_0} - \epsilon^2 k^2 \right\}. \tag{42}$$

For (7):

$$\sigma(k) = k^2(\epsilon^2 k^2 + q'(\rho_0)). \tag{43}$$

These three dispersion relations are qualitatively similar, in that in all cases,

$$\sigma(0) = 0, \quad \mathrm{Re}\ \sigma(k) > 0 \text{ for } 0 < \epsilon k < \gamma a, \quad \mathrm{Re}\ \sigma(k) < 0 \text{ for } \epsilon k > \gamma a, \tag{44}$$

where the constant γ is 1, $(A\rho_0)^{-1/2}$, and $\rho_0^{-1/2}$ in the three cases. This means that the constant solution is unstable to a finite range of wave numbers $k = O(\epsilon^{-1})$, and stable for k larger than that range. There is in each case a particular value of k for which σ is maximal: disturbances with wave number at or near this value grow the fastest.

We can expect, in view of this maximal growth rate, that the subsequent evolution of solutions starting at a random small enough perturbation will retain the dominance of the wave number giving the maximal σ into the nonlinear regime, and approach some neighborhood of a periodic traveling wave solution with that wave number. Results similar to this were proved for spinodal decomposition problems for the Cahn-Hilliard equation in [3] and (in 2D) [9].

It is our conjecture that solutions starting with the periodicity and symmetry of our constructed traveling wave solutions and with average density in certain range evolve to the appropriate periodic solution obtained in the previous sections, and moreover that the spinodal decomposition process occurs for the regularized systems we have studied in much the same manner as it does for the Cahn-Hilliard equation.

5 Discussion

We have given a relatively simple existence proof for periodic traveling wave solutions of (3), (4), as well as of (5), (6), of period 1 when ϵ is smaller than an explicit number λ_m^{-1} and when the prescribed average density lies in an explicit interval. It has layer transitions when ϵ is very small. We have also discussed how these patterned solutions probably evolve out of unstructured initial data, as in "spinodal decomposition" for the Cahn-Hilliard equation. It would be interesting to see whether such periodic structures can be observed experimentally in real fluids when the van der Waals model may be relevant.

Regarding the existence results, since changing ϵ really amounts to changing the space scale, we may in fact choose the period to be any positive number. We thus obtain layered traveling wave solutions of any wavelength. The small parameter is the same in the two equations (3) and (4), but they could be taken different (ϵ and $\mu\epsilon$, say, with μ a fixed positive number), and our conclusions would be essentially the same.

If we wish solutions with average value of ρ outside the given interval, then of course there are constant single-phase solutions of (10), (11) with that property. However, it is unlikely that nonconstant solutions exist.

As $\epsilon \to 0$, our layered solutions approach discontinuous solutions of (1), (2) which satisfy the standard shock conditions. It is a system which changes type. There are many other discontinuous traveling wave solutions with the same properties, for example ones which take on the same two values but are not periodic.

Although our object of study is the existence of traveling waves with given wavelength, ϵ, and average values, we can also obtain (in a much simpler way) the existence of traveling front solutions which correspond to infinite wavelength. The front profiles are simply the heterclinic orbits mentioned in Thm. 1(a). In these fronts, the function ρ will be monotone. Grinfeld [4] studied front solutions for a similar problem, and obtained nonmonotone as well as monotone profiles. He termed the former "cavitation waves" since the density takes on a minimum in the transition region. Such nonmonotone waves also exist in the present case; they correspond to nonzero values of the constant $\bar{a}$ in (12) (so the system would not be Hamiltonian); we have not considered them explicitly.

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