EXISTENCE OF HETEROCLINIC ORBITS FOR A CORNER LAYER PROBLEM IN ANISOTROPIC INTERFACES

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Abstract. Mathematically, the problem considered here is that of heteroclinic connections for a system of two second order differential equations of gradient type, in which a small parameter ϵ conveys a singular perturbation. The physical motivation comes from a multi-order-parameter phase field model developed by Braun et al [BCMcFW] and [T] for the description of crystalline interphase boundaries. In this, the smallness of ϵ is related to large anisotropy. The mathematical interest stems from the fact that the smoothness and normal hyperbolicity of the critical manifold fails at certain points. Thus the well-developed geometric singular perturbation theory [Fe], [J] does not apply. The existence of such a heteroclinic, and its dependence on ϵ, is proved via a functional analytic approach.

1. Introduction

We consider the problem of finding heteroclinic solutions $x(s), y(s)$, of the singularly perturbed system

$$\begin{array}{rcl} x'' & = & g_x(x,y) \\ \epsilon^2 y'' & = & g_y(x,y) \end{array} \tag{1.1}$$

which are approximated, when ϵ is small, by a nonsmooth connection for the formal limiting system obtained by setting $\epsilon = 0$. The irregularity in the formal limit arises due to the branching nature of the set of solutions (x, y) of the degenerate relation $0 = g_y(x, y)$ (see Figure 2). Thus the critical manifold will have a "corner" singularity. The salient features of the function $g \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ are

(i) $g(x, y) = g(x, -y), \quad \forall (x, y) \in \mathbb{R}^2$. (This symmetry is assumed for convenience only.)

(ii) g has three nondegenerate equal global minima at $(0, 0)$, (x_1, y_1), and $(x_1, -y_1)$, where $x_1 > 0$, and $y_1 > 0$. Without loss of generality, we assume that $g(0, 0) = g(x_1, y_1) = g(x_1, -y_1) = 0$.

(iii) There exists a continuous function, with finitely many points of nondifferentiability $Y : [-\delta, x_1 + \delta] \to [0, \infty)$, $\delta > 0$ small, such that $Y(0) = 0$, $Y(x_1) = y_1$, $g_y(x, Y(x)) = 0$, $\forall x \in [-\delta, x_1 + \delta]$ and $g(x, Y(x)) \neq 0$, $\forall x \in (0, x_1)$.

We are interested in solutions $(x, y) \in C^2(\mathbb{R}) \times C^2(\mathbb{R})$ of (1.1) which satisfy

$$(x(-\infty), y(-\infty)) = (0, 0) \quad \text{and} \quad (x(\infty), y(\infty)) = (x_1, y_1) \tag{1.2}$$

and their projection on the $x-y$ plane converges as $\epsilon \to 0$ to the set $\{(x, Y(x)),\ x \in (0, x_1)\}$. From the hypotheses on g one can easily show that there exists a solution $(x_0, y_0) \in C^2(\mathbb{R}) \cap C(\mathbb{R})$ with $y_0(\cdot) = Y(x_0(\cdot))$ and $x_0' > 0$ of (1.1), (1.2) for $\epsilon = 0$ (cf. Section 3).

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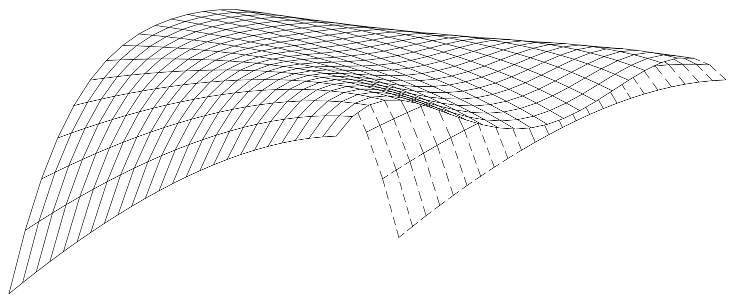

FIGURE 1. The graph of $-g$

In this paper we study (1.1), (1.2) for a specific function g satisfying (i), (ii), (iii) such that there exists an $x_c \in (0, x_1)$ with

$$g_{yy}(x_c, Y(x_c)) = 0, \quad g_{yy}(x, Y(x)) > 0, \; x \neq x_c \tag{1.3}$$

and

$$Y \in C(-\delta, x_1 + \delta) \cap C^\infty((-\delta, x_1 + \delta) - \{x_c\}) \text{ is not differentiable at } x = x_c. \tag{1.4}$$

More specifically g is given by

$$g(x, y) = 2(x^2 + y^2) - 6xy^2 + x^2y^2 + \frac{3}{4}y^4 + x^4. \tag{1.5}$$

The physical motivation for this choice of g is given later in the introduction. We remark that we believe that our approach works for a more general class of functions g satisfying (1.3) and having the same type of local behaviour near the bifurcation point x_c.

We have (cf. Section 2):

$$Y(x) = \begin{cases} 0, & x \leq x_c = 3 - \sqrt{7} \cong 0.35 \;\text{ with }\; 2 - 6x_c + x_c^2 = 0 \\ \sqrt{\frac{2}{3}}(-x^2 + 6x - 2)^{\frac{1}{2}}, & x_c \leq x \leq 3 + \sqrt{7} \end{cases}$$

and $(x_1, y_1) = (1, \sqrt{2})$. Thus we consider (1.1) together with the conditions

$$(x(-\infty), y(-\infty)) = (0, 0) \quad \text{and} \quad (x(\infty), y(\infty)) = (1, \sqrt{2}). \tag{1.6}$$

The graph of $-g$ is given in Figure 1 and the level set $g_y = 0$ is as in Figure 2.

Let us return to (1.1), (1.2) for a general g satisfying (i), (ii), (iii). By setting $u_1 = x, \; u_2 = x', \; u_3 = y, \; u_4 = \epsilon y'$, we can write (1.1) in the form

$$\begin{array}{rcl} u_1' &=& u_2 \\ u_2' &=& g_x(u_1, u_3) \\ \epsilon u_3' &=& u_4 \\ \epsilon u_4' &=& g_y(u_1, u_3) \end{array} \tag{1.7}$$

For $\epsilon = 0$ we get

$$\begin{array}{rcl} u_1' &=& u_2 \\ u_2' &=& g_x(u_1, u_3) \\ 0 &=& u_4 \\ 0 &=& g_y(u_1, u_3) \end{array} \tag{1.8}$$

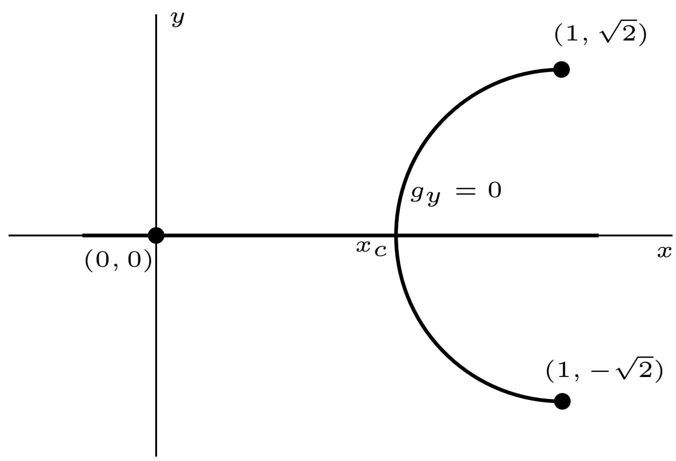

FIGURE 2. The form of the set $g_y(x, y) = 0$

The last two equations of (1.8) define a two dimensional manifold $M_0 = \{u_3 = Y(u_1),\ u_4 = 0,\ (u_1, u_2) \in K\}$ where $K \subset \mathbb{R}^2$ is compact and int$(K)\supset \{(x_0(s), x_0'(s)),\ s \in \mathbb{R}\}$. The first two equations of (1.8) define a nontrivial flow on M_0 (cf. [J]). Based on the theory presented in [J], for studying the behaviour of solutions of (1.7) close to M_0 for $\epsilon > 0$ small, we examine the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ g_{xy}(u_1, Y(u_1)) & 0 & g_{yy}(u_1, Y(u_1)) & 0 \end{bmatrix}$$

i.e

$$\lambda^2[\lambda^2 - g_{yy}(u_1, Y(u_1))] = 0.$$

If $g_{yy}(x, Y(x)) > 0,\ x \in [-\delta, x_1 + \delta]$ and $Y \in C^2(-\delta, x_1 + \delta)$ then we have only two eigenvalues on the imaginary axis and since $M_0 \in C^2$ we conclude that M_0 is smooth and normally hyperbolic. The problem in this case is rather well understood using the invariant manifold approach of geometric singular perturbation theory [Fe], [J]. Actually under the above hypotheses, there exists a solution (x_ϵ, y_ϵ) of (1.1), (1.2) with $\epsilon > 0$ small, such that $(x_\epsilon, y_\epsilon) \overset{\epsilon \to 0}{\to} (x_0, y_0)$ uniformly in $\mathbb{R}$. This was proved in [AFFS2] using geometric singular perturbation theory and in [AFFS1] using a functional analytic approach which also gives information on the spectrum of the linearized operator (stability).

The geometric singular perturbation theory as presented in [J] cannot be applied if g is as in (1.5). Indeed, from (1.3), (1.4) we see that the "singular solution" (cf. [J]) $\{(x_0(s), x_0'(s), Y(x_0(s)), 0),\ s \in \mathbb{R}\} \subset M_0$ passes through a non smooth, non hyperbolic point.

The specific choice of g is motivated from [BCMcFW] where a continuum model was derived for studying interfaces in the context of crystals. It is based on the free energy functional

$$J(X, Y, Z) = \int_\Omega [Q(\nabla X, \nabla Y, \nabla Z) + F(X, Y, Z)]\, d\xi_1 d\xi_2 d\xi_3, \tag{1.9}$$

where X, Y, Z are composition variables, (ξ_1, ξ_2, ξ_3) space coordinates and Ω the volume occupied by the sample. Here Q is a positive definite quadratic form and F is positive except at its several global minima. The bulk free energy F must conform to certain crystalline symmetries, for instance it must be invariant under permutation of X, Y, Z. If it is restricted to be a fourth degree polynomial its general form is that given below in (1.10).

The function Q represents the influence on the free energy of the difference between the order parameters at a crystalline vertex and those at nearest neighbors. It is generally the case that this contribution depends on the orientation of the line between those nearby points, and this dependence is a source of anisotropy. The simplest type of quadratic form Q which accounts for anisotropy and which satisfies other symmetry conditions is of the form $Q = AQ_1 + BQ_2$ where Q_i are the simple sums of squares of derivatives given below in (1.11). The ratio $B/A := \epsilon^2$ then will be taken as a measure of the degree of anisotropy of the free energy. Isotropy corresponds to the case $B = A$ as we show below ($\epsilon = 1$); our focus will be on the anisotropic case $\epsilon \ll 1$. The simplest example for F and Q is

$$F(X,Y,Z) = a_2(X^2+Y^2+Z^2)+a_3XYZ+a_{41}(X^4+Y^4+Z^4)+a_{42}(X^2Y^2+X^2Z^2+Y^2Z^2), \tag{1.10}$$

$$\begin{aligned} Q_1 &= \frac{1}{2}\left[\left(\frac{\partial X}{\partial \xi_1}\right)^2 + \left(\frac{\partial Y}{\partial \xi_2}\right)^2 + \left(\frac{\partial Z}{\partial \xi_3}\right)^2\right], \\ Q_2 &= \frac{1}{2}\left[\left(\frac{\partial X}{\partial \xi_2}\right)^2 + \left(\frac{\partial X}{\partial \xi_3}\right)^2 + \left(\frac{\partial Y}{\partial \xi_1}\right)^2 + \left(\frac{\partial Y}{\partial \xi_3}\right)^2 + \left(\frac{\partial Z}{\partial \xi_1}\right)^2 + \left(\frac{\partial Z}{\partial \xi_2}\right)^2\right]. \end{aligned} \tag{1.11}$$

The approach in [BCMcFW] is to assume dynamics governed by a gradient flow with respect to J, and examine the nature of the interface between grains of ordered and disordered material. In general, these two states will enjoy different bulk free energies, and the interface will migrate. However, the motion depends on the values of the coefficients in (1.10), which in turn depend on the temperature. The simplest situation is when the two bulk values of F are the same. In this case we are led as we will see below to a hamiltonian system. In the specific example (1.10) if the temperature is chosen appropriately then this is the case, with the two equilibria $(0, 0, 0)$ and $(1, 1, 1)$ representing the disordered and ordered state respectively. A possible choice of the coefficients in (1.10) such that $F(0, 0, 0) = F(1, 1, 1) = 0$ is

$$a_2 = 2, \ a_3 = -12, \ a_{41} = a_{42} = 1 \tag{1.12}$$

(see [BCMcFW]). The governing evolution PDE's are given by the L^2 gradient flow of this functional:

$$\tau \frac{\partial}{\partial t}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = L\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} - \nabla F(X,Y,Z), \tag{1.13}$$

where L is a diagonal matrix of second degree elliptic operators in the space variables, ∇ denotes the gradient with respect to the variables (X, Y, Z), and τ is a dimensionless relaxation time.

Plane waves in the direction $\bar{n}$ and velocity V are solutions of (1.13) of the form

$$X = x(\bar{n} \cdot (\xi_1, \xi_2, \xi_3) - Vt) = x(s), \quad Y = y(s), \quad Z = z(s),$$

with boundary conditions

$$x(-\infty) = y(-\infty) = z(-\infty) = 0, \quad x(\infty) = y(\infty) = z(\infty) = 1. \tag{1.14}$$

They represent planar interfaces with normal $\bar{n}$ separating an ordered state from a disordered state. The functions $\bar{u} = (x, y, z)$ satisfy (derivatives are with respect to s):

$$-V\bar{u}' = \Lambda\bar{u}'' - \nabla F(x, y, z), \tag{1.15}$$

where $\Lambda = [\Lambda_{ij}]$ is a diagonal matrix whose elements are linear functions of A and B and quadratic functions of $\bar{n}$. However recall that the coefficients of F are such that F has equal depth wells at the equilibria at the order disorder transition: $F(0,0,0) = F(1,1,1)$. We see by taking the scalar product of (1.15) with $\bar{u}'$ and integrating that $V = 0$. The resulting system is hamiltonian.

When $\bar{n} = (1,0,0)$ then $\Lambda_{11} = A$ and $\Lambda_{22} = \Lambda_{33} = B$ (cf. [BCMcFW]). The above suggest that we seek solutions of (1.15) with $V = 0$ with the imposed symmetry condition $y = z$. For exhibiting the resulting system we define our anisotropy parameter $\epsilon := \sqrt{B/A}$ and the reduced free energy

$$g(x, y) = F\left(x, \frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}y\right)$$

which corresponds, via (1.10), to (1.5) for the choice of coefficients given in (1.12). We have that the resulting system, is modulo rescaling, (1.1) together with (1.6). For more details we refer the interested reader to [BCMcFW], [ABCFFT].

Throughout the rest of the paper we consider (1.1), (1.6) with g given by (1.5). The main result of the paper is

Theorem. *If $\epsilon > 0$ is sufficiently small, then there exists a solution $(x_\epsilon, y_\epsilon) \in C^2(\mathbb{R}) \times C^2(\mathbb{R})$ of (1.1), (1.6) with*

$$||x_\epsilon - x_0||_{H^2(\mathbb{R})} + \epsilon^{\frac{1}{3}}||y_\epsilon - y_0||_{L^2(\mathbb{R})} + \epsilon^{\frac{2}{3}}||y_\epsilon - y_0||_{L^\infty(\mathbb{R})} \le C\epsilon,$$

where (x_0, y_0) solves (1.1) with $\epsilon = 0$ and (1.6).

We give a brief overview of the structure of the paper which is devoted to the proof of the above Theorem. In Section 2 we show that this g satisfies (i), (ii), (iii) with $(x_1, y_1) = (1, \sqrt{2})$ and we find the explicit function $Y(\cdot)$ such that $g_y(x, Y(x)) = 0,\ x \in \text{Domain}(Y)$. We have that there exists an $x_c \in (0,1)$ such that $Y \in C((-\infty, 3]) \cap C^\infty((-\infty, 3] - \{x_c\})$, $Y(x) = 0,\ x \le x_c$, $Y'(x) \approx C(x - x_c)^{-\frac{1}{2}}$ for $x - x_c > 0$ sufficiently small, and $Y'(x) > 0,\ x \in (x_c, 3]$. In Section 3 we show the existence of a solution $(x_0, y_0) = (x_0, Y(x_0)) \in C^2(\mathbb{R}) \times C(\mathbb{R})$ of (1.1) with $\epsilon = 0$ and (1.6). We assume due to translation invariance that $x_0(0) = x_c$ and thus y_0 is not smooth at $s = 0$. The pair (x_0, y_0) corresponds to the outer solution of (1.1), (1.6) for small $\epsilon > 0$, i.e it is a good approximation of the true solution outside of a neighborhood of $s = 0$ (note also that $y_0' \to \infty$ as $s \to 0^+$). We seek an approximate solution valid for $s \in \mathbb{R}$ in the form (x_0, Y_r^ϵ) where $Y_r^\epsilon \in C^2(\mathbb{R})$ solves the "reduced" problem

$$\epsilon^2 y'' = g_y(x_0, y), \qquad s \in \mathbb{R} \tag{1.16}$$

$$y(-\infty) = 0, \qquad y(\infty) = \sqrt{2} \tag{1.17}$$

and $Y_r^\epsilon \overset{L^\infty(\mathbb{R})}{\to} y_0$ as $\epsilon \to 0$ (recall that $g_y(x_0, y_0) = 0$). The main difficulties for proving existence of such a function Y_r^ϵ are that $y_0 \in C(\mathbb{R})$ is not smooth at $s = 0$

and $g_{yy}(x_0(0), y_0(0)) = 0$. We feel that it is useful to graph the form of the function $g_{yy}(x_0(s), y_0(s)),\ s \in \mathbb{R}$ in Figure 3.

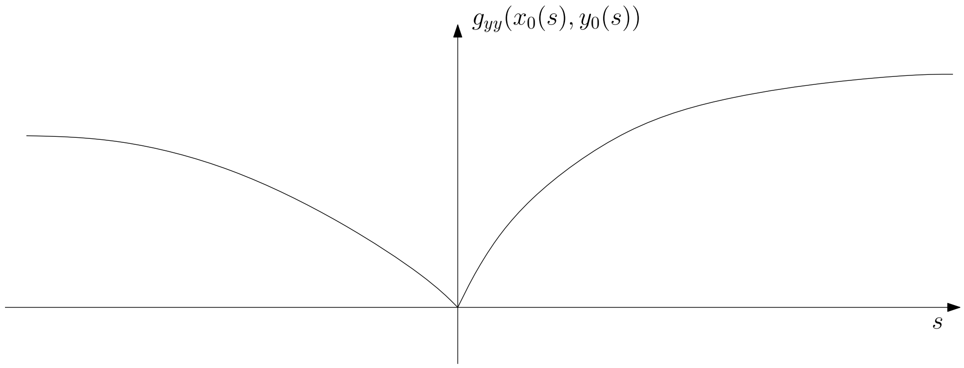

FIGURE 3. The form of the function $g_{yy}(x_0(\cdot), y_0(\cdot))$

In Section 4 we construct a $Y_{ap}^\epsilon \in C^1(\mathbb{R}) \cap C^\infty(\mathbb{R} - \{\pm|\ln \epsilon|\epsilon^{\frac{2}{3}}\})$ which solves (1.16) approximately in $\mathbb{R}$ and (1.17) exactly. We estimate $|\epsilon^2 {Y_{ap}^\epsilon}'' - g_y(x_0, Y_{ap}^\epsilon)|$ both in $L^\infty(\mathbb{R})$ and in $L^2(\mathbb{R})$; the latter will be used for studying the system (1.1). In Section 5 we consider the linear operator

$$L_\epsilon y = -\epsilon^2 y'' + g_{yy}(x_0, \tilde{Y}^\epsilon) y, \quad y \in H^2(\mathbb{R}) \tag{1.18}$$

for a class of functions $\tilde{Y}^\epsilon$ which includes Y_{ap}^ϵ. We remark that $g_{yy}(x_0, \tilde{Y}^\epsilon)$ can be negative for $|s| = O(\epsilon^{\frac{2}{3}})$ and thus we have not been able to use the results of [FP]. Instead, we establish a priori estimates for the equations $L_\epsilon y = f$ and $L_\epsilon y = \tilde{Y} f$ using blow up (scaling) arguments. In the first case we get that $||y||_{L^p(\mathbb{R})} \le C\epsilon^{-\frac{2}{3}}||f||_{L^p(\mathbb{R})}$ and in the second $||y||_{L^p(\mathbb{R})} \le C\epsilon^{-\frac{1}{3}}||f||_{L^p(\mathbb{R})}$ with $p = 2, \infty$. We seek a solution of (1.16), (1.17) in the form $Y_{ap}^\epsilon + y$ with $y \in H^2(\mathbb{R})$ (recall that functions in $H^1(\mathbb{R})$ tend to 0 at $\pm\infty$). Substituting in (1.16) and rearranging terms yields

$$L_\epsilon y = c_1 y^3 + c_2 Y_{ap}^\epsilon y^2 + \epsilon^2 {Y_{ap}^\epsilon}'' - g_y(x_0, Y_{ap}^\epsilon)$$

for some $c_i \in \mathbb{R}$ and L_ϵ is as in (1.18) with $\tilde{Y}^\epsilon = Y_{ap}^\epsilon$. In Section 6 we prove existence of a solution of the above equation using the contraction mapping principle. This solution furnishes a solution $Y_r^\epsilon \in C^2(\mathbb{R})$ of (1.16), (1.17) such that $||Y_r^\epsilon - y_0||_{L^\infty(\mathbb{R})} \le C\epsilon^{\frac{1}{3}}$ and $||Y_r^\epsilon - y_0||_{L^2(\mathbb{R})} \le C\epsilon^{\frac{2}{3}}$. We remark that the results of Section 5 hold for L_ϵ with $\tilde{Y}^\epsilon = Y_r^\epsilon$. Next we seek a solution of (1.1), (1.6) in the form $(x_0 + x, Y_r^\epsilon + y)$ with $x, y \in H^2(\mathbb{R})$. Substituting in (1.1) and rearranging terms yields

$$-Bx = O(x^2 + y^2) + g_{xy}(x_0, y_0)\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)} x + y\right) + E_1(x, y)$$

$$-L_\epsilon y = O(xy + x^2 + Y_r^\epsilon y^2 + y^3) + g_{xy}(x_0, Y_r^\epsilon) x$$

The linear operator $Bx = -x'' + \left(g_{xx}(x_0, y_0) - \frac{g_{xy}^2(x_0,y_0)}{g_{yy}(x_0,y_0)}\right) x,\ \ x \in H^2(\mathbb{R})$ is studied in Section 8. It holds that $g_{xy}^2(x_0, y_0) \le C g_{yy}(x_0, y_0),\ s \in \mathbb{R}$ for some $C > 0$. Hence B is selfadjoint in $L^2(\mathbb{R})$. Moreover we have that $\sigma(B) \subset \{0\} \cup [c, \infty)$ for some

$c > 0$ and 0 is a simple eigenvalue. This is well known since B corresponds to the variational equation at x_0 of (3.4) below, however we provide an elementary proof of this. The linear operator L_ϵ is as in (1.18) with $\tilde{Y}^\epsilon = Y_r^\epsilon$ and $||E_1(x,y)||_{L^2(\mathbb{R})} \leq C\epsilon + C\epsilon^{\frac{2}{3}}||y||_{L^\infty(\mathbb{R})}$. In Section 7 we prove that

$$L_\epsilon y = -\epsilon^2 y'' + g_{yy}(x_0, Y_r^\epsilon)y = -g_{xy}(x_0, Y_r^\epsilon)x, \quad x \in H^2(\mathbb{R})$$

implies

$$\left|\left| g_{xy}(x_0,y_0)\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x + y\right)\right|\right|_{L^2(\mathbb{R})} \leq C\epsilon^{\frac{1}{6}}||x||_{H^2(\mathbb{R})}.$$

Combining the above with the gradient structure of (1.1) motivates us to consider a map

$$\mathbf{T} : \{(x,y) \in H^2(\mathbb{R}) \times H^2(\mathbb{R}),\ x \perp x_0'\} \to \{(x,y) \in H^2(\mathbb{R}) \times H^2(\mathbb{R}),\ x \perp x_0'\}$$

($x \perp x_0'$ means in $L^2(\mathbb{R})$) via $\mathbf{T}(x,y) = (\bar{x}, \bar{y})$, where

$$-B\bar{x} = -b(x,y)x_0' + O(x^2+y^2) + g_{xy}(x_0,y_0)\left(\tfrac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x + \bar{y}\right) + E_1(x,y)$$

$$-L_\epsilon \bar{y} = O(xy + x^2 + Y_r^\epsilon y^2 + y^3) + g_{xy}(x_0, Y_r^\epsilon)x$$

and $b(x,y)$ is a number such that the right hand side of the first equation is $\perp x_0'$ which allows us to solve for $\bar{x}$. In Section 9 we prove existence of a fixed point of $\mathbf{T}$, which gives the existence of $(x_\epsilon, y_\epsilon) \in C^2(\mathbb{R}) \times C^2(\mathbb{R})$ satisfying (1.6) and a $b_\epsilon \in \mathbb{R}$ such that

$$\begin{aligned} x_\epsilon'' &= g_x(x_\epsilon, y_\epsilon) - b_\epsilon x_0' \\ \epsilon^2 y_\epsilon'' &= g_y(x_\epsilon, y_\epsilon) \end{aligned}$$

Furthermore,

$$||x_\epsilon - x_0||_{H^2(\mathbb{R})} + \epsilon^{\frac{1}{3}}||y_\epsilon - y_0||_{L^2(\mathbb{R})} + \epsilon^{\frac{2}{3}}||y_\epsilon - y_0||_{L^\infty(\mathbb{R})} \leq C\epsilon.$$

From the above two relations and the fact that $g(0,0) = g(1,\sqrt{2})$, it follows that $b_\epsilon = 0$, i.e (x_ϵ, y_ϵ) solves (1.1), (1.6) and satisfies the estimate of the Theorem.

Our approach is functional analytic in the spirit of [AFFS1] and uses as its point of departure [ABCFFT]. In the latter reference, the leading order interior approximation to a similar equation as in (1.16) was established (cf. Propositions 4.1, 5.1).

A similar result to our main Theorem appears in [Fi]. There a shooting type-argument is used. This produces a solution (x_ϵ, y_ϵ) of (1.1), (1.6) (with g given by (1.5)) that converges to (x_0, y_0) as $\epsilon \to 0$, although less information about this convergence was obtained.

We must say that problem (1.1) considered in a bounded interval and without the right hand side having gradient form has been studied extensively by a variety of methods. We refer to [TKJ] for a review and on how many cases are handled using geometric singular perturbation theory. Note that via (ii) we have that for $\epsilon > 0$ both $(0,0,0,0)$ and $(x_1, 0, y_1, 0)$ are saddles for (1.7) with two-dimensional stable and unstable manifolds. Thus their intersection is not transversal. Based on this, we believe that in our case the gradient structure is necessary. For recent extensions of geometric singular perturbation theory to deal with certain types of nonhyperbolic points in the plane, we refer to [KS].

2. Formulation of the problem

We will study the existence of solutions with $\epsilon > 0$ small for the following problem:

$$\begin{array}{rcl} x'' & = & g_x(x,y) \\ \epsilon^2 y'' & = & g_y(x,y) \end{array} \tag{2.1}$$

$$x = x(s), \quad y = y(s), \quad s \in \mathbb{R}, \quad ' = \frac{d}{ds}$$

together with the boundary conditions

$$x(-\infty) = 0, \qquad x(\infty) = 1, \tag{2.2}$$

$$y(-\infty) = 0, \qquad y(\infty) = \sqrt{2}, \tag{2.3}$$

where

$$g(x,y) = 2(x^2+y^2) - 6xy^2 + x^2y^2 + \frac{3}{4}y^4 + x^4. \tag{2.4}$$

For future reference, note

$$g_x(x,y) = 4x - 6y^2 + 2xy^2 + 4x^3, \quad g_y(x,y) = y(4 - 12x + 2x^2 + 3y^2), \tag{2.5}$$

$$\begin{array}{rcl} g_{xx}(x,y) & = & 4 + 2y^2 + 12x^2 \geq 4, \\ g_{xy}(x,y) & = & -12y + 4xy, \\ g_{yy}(x,y) & = & 4 - 12x + 2x^2 + 9y^2. \end{array} \tag{2.6}$$

We will find the critical points of g, i.e. solve $g_x(x,y) = 0$ and $g_y(x,y) = 0$. From $g_y = 0$ we get $y = 0$ or $y = \pm Y(x)$ where

$$Y(x) = \begin{cases} 0, & x \leq x_c = 3 - \sqrt{7} \cong 0.35 \ \text{ with } \ 2 - 6x_c + x_c^2 = 0, \\ \sqrt{\frac{2}{3}}(-x^2 + 6x - 2)^{\frac{1}{2}}, & x_c \leq x \leq 3 + \sqrt{7}. \end{cases} \tag{2.7}$$

By substituting in $g_x = 0$ we get as the only critical points:

$$\begin{cases} (0,0)\ ,\ (1, \pm\sqrt{2}) & \text{global nondegenerate minima of } g, \\ \left(\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right) & \text{saddle points.} \end{cases} \tag{2.8}$$

Note that $0 < x_c < \frac{1}{2}$. Also

$$g(0,0) = g(1,\sqrt{2}) = g(1,-\sqrt{2}) = 0. \tag{2.9}$$

Furthermore g is symmetric with respect to the x axis. The above imply that g satisfies (i), (ii), (iii) of the introduction.

Also by setting $a = \frac{3}{2}$, $b = 6 - 2x_c > 0$, we observe that

$$g_y(x,y) = 2y\left(ay^2 - b(x - x_c) + (x - x_c)^2\right). \tag{2.10}$$

Remark 2.1. *In this paper, unless specified otherwise, we will denote by C/c a large/small positive generic constant independent of ϵ whose value will change from line to line. In many cases we will not explicitly write the obvious dependence of functions on $\epsilon > 0$.*

3. The $\epsilon = 0$ approximation (x_0, y_0)

Setting $\epsilon = 0$ in (2.1), yields

$$\begin{aligned} x'' &= g_x(x,y) \\ 0 &= g_y(x,y). \end{aligned} \tag{3.1}$$

From the second equation of (3.1) we get

$$y = Y(x) \tag{3.2}$$

and by substituting in the first

$$x'' = g_x(x, Y(x)) \tag{3.3}$$

and since $g_y(x, Y(x)) = 0$, we have

$$x'' = G_x(x) \tag{3.4}$$

where

$$G(x) = g(x, Y(x)), \quad x \leq 3. \tag{3.5}$$

We note that $G \in C^1(-\infty, 3]$ and $G_{xx} \in C\left((-\infty, 3] - \{x_c\}\right)$ with a finite jump discontinuity at $x = x_c$ (this follows from (2.5), (2.6)). From (2.8), (2.9) and since $Y(0) = 0$, $Y(1) = \sqrt{2}$, we get

$$\begin{array}{cc} G(0) = 0 & G(1) = 0 \\ G_x(0) = 0 & G_x(1) = 0 \\ G_{xx}(0) > 0 & G_{xx}(1) > 0 \end{array} \tag{3.6}$$

$$G(x) \neq 0, \quad x \in (0,1). \tag{3.7}$$

From (3.6), (3.7) we obtain that (3.4) and thus (3.3) has a unique solution $x_0(\cdot)$ such that

$$x_0(-\infty) = 0, \quad x_0(\infty) = 1, \quad \text{and} \quad x_0(0) = x_c \quad \text{(cf. [Ar])}. \tag{3.8}$$

Furthermore

$$x_0'(s) > 0, \quad s \in \mathbb{R}. \tag{3.9}$$

Also x_0 approaches its limits exponentially and

$$x_0', \ x_0'', \ x_0''' \to 0 \quad \text{exponentially as} \ \ |s| \to \infty. \tag{3.10}$$

Note that (3.10) combined with the fact that x_0''' has a finite jump discontinuity only at $s = 0$ gives that $x_0' \in H^2(\mathbb{R})$.
From (3.2) we have $y_0(s) := Y(x_0(s))$, $s \in \mathbb{R}$.
We observe that $y_0 \in C(\mathbb{R}) \cap C^\infty(\mathbb{R} - \{0\})$ and via (2.7), (3.8), (3.9):

$$\frac{3}{2} y_0^2 = -(x_0 - x_c)(x_0 + x_c - 6), \quad s \geq 0,$$

$y_0(s) > 0, \ s > 0$ and $3y_0 y_0' = (-2x_0 + 6)x_0', \ s > 0.$ We see from (3.8), (3.9), (3.10) and the above that

$$y_0'(s) > 0, \quad s > 0, \tag{3.11}$$

$$y_0(s) = 0, \quad s \leq 0, \qquad y_0(\infty) = \sqrt{2} \ \ \text{(exponentially)}, \tag{3.12}$$

$$y_0', \ y_0'' \to 0 \ \ \text{as} \ \ s \to \infty \ \ \text{(exponentially)}. \tag{3.13}$$

Furthermore since $x_0(0) = x_c$,

$$\lim_{s\to 0^+} \frac{y_0^2(s)}{s} = C > 0, \text{ i.e. } \lim_{s\to 0^+} \frac{y_0(s)}{s^{\frac{1}{2}}} = C > 0.$$

This and the relation

$$s^{\frac{1}{2}} y_0' = \frac{1}{3}(-2x_0 + 6)x_0' \frac{s^{\frac{1}{2}}}{y_0(s)}, \quad s > 0,$$

yield $\lim_{s\to 0^+} s^{\frac{1}{2}} y_0'(s) = C > 0$. Similarly we obtain $\lim_{s\to 0^+} s^{\frac{3}{2}} y_0''(s) = C \in \mathbb{R}$. The above three limits combined with (3.13), imply

$$y_0(s) \le Cs^{\frac{1}{2}}, \quad y_0'(s) \le Cs^{-\frac{1}{2}}, \quad |y_0''(s)| \le Cs^{-\frac{3}{2}}, \quad s > 0, \tag{3.14}$$

$$cs^{\frac{1}{2}} \le y_0(s), \quad 0 \le s \le 1. \tag{3.15}$$

The pair (x_0, y_0) satisfies (3.1) and (2.2), (2.3). The approximation (x_0, y_0) is not satisfactory due to the serious lack of smoothness of y_0. We keep x_0, but we will replace y_0 by $Y_r^\epsilon \in C^2(\mathbb{R})$ which solves the "reduced" problem:

$$\epsilon^2 y'' = g_y(x_0, y), \quad s \in \mathbb{R} \tag{3.16}$$

$$y(-\infty) = 0, \quad y(\infty) = \sqrt{2}. \tag{3.17}$$

4. Construction of an approximate solution Y_{ap}^ϵ for the "reduced" problem (3.16), (3.17)

In this section we will construct a C^1 approximate solution Y_{ap}^ϵ for the reduced problem (3.16), (3.17). We first construct a $y_{ap}^\epsilon \in C(\mathbb{R})$ as

$$y_{ap}^\epsilon = \begin{cases} y_{out}^-, & s \le -|\ln\epsilon|\epsilon^{\frac{2}{3}} \\ y_{in}, & |s| \le |\ln\epsilon|\epsilon^{\frac{2}{3}} \\ y_{out}^+, & s \ge |\ln\epsilon|\epsilon^{\frac{2}{3}} \end{cases} \tag{4.1}$$

where y_{out}^-, y_{in}, $y_{out}^+ \in C^\infty$ satisfy (3.16) approximately in their domain of definition, and y_{out}^-, $y_{out}^+ \cong y_0$ satisfy (3.17). We then obtain Y_{ap} as

$$Y_{ap} := y_{ap} + \hat{\rho} \tag{4.2}$$

where $\hat{\rho} \in H^1(\mathbb{R})$ a suitable ("small") function such that $Y_{ap} \in C^1(\mathbb{R})$ and Y_{ap} satisfies (weakly) "approximately" (3.16) in $\mathbb{R}$. We note that $\hat{\rho} \in H^1(\mathbb{R})$ implies that Y_{ap} satisfies (3.17).

Throughout this paper we will use the notation $d_\epsilon = |\ln\epsilon|\epsilon^{\frac{2}{3}}$.

<u>The inner approximation y_{in}^ϵ</u>

From (2.10) with $m = x_0'(0) > 0$ and $y \in C^2(\mathbb{R})$:

$$-\epsilon^2 y'' + g_y(x_0, y) = -\epsilon^2 y'' + 2y(ay^2 - bms) + yF(s), \ s \in \mathbb{R}. \tag{4.3}$$

with $|F(s)| \le Cs^2$, $s \in \mathbb{R}$ ($C > 0$ independent of y). We will find $y_{in}(s)$ as a suitable solution of

$$\epsilon^2 y'' = 2y(ay^2 - bms), \ s \in \mathbb{R} \tag{4.4}$$

and then we will restrict its domain of definition to $|s| \leq d_\epsilon$. Using the transformation

$$y(s) = \epsilon^{\frac{1}{3}} R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) \tag{4.5}$$

we see that

$$R''(s) = 2R(s)(aR^2(s) - bms), \ s \in \mathbb{R}. \tag{4.6}$$

We want $\epsilon^{-\frac{1}{3}}\left[y_{in}(\pm d_\epsilon) - y_0(\pm d_\epsilon)\right] \to 0$ as $\epsilon \to 0$. Since $y_0(s) = 0$ for $s \leq 0$, and

$$\left| y_0(s) - \sqrt{\frac{bm}{a}s} \right| \leq C s^{\frac{3}{2}}, \quad s \geq 0, \tag{4.7}$$

we have that the appropriate conditions on R are

$$R(s) \to 0, \quad s \to -\infty, \tag{4.8}$$

$$R(s) - \sqrt{\frac{bm}{a}s} \to 0, \quad s \to +\infty, \tag{4.9}$$

(see also (4.19)).
We have

Proposition 4.1. *There exists a solution $R(s)$ of (4.6), with $R'(s) > 0$, $s \in \mathbb{R}$ and*

$$R(s) = O(e^s) \quad as \quad s \to -\infty, \tag{4.10}$$

$$R(s) = \sqrt{\frac{bm}{a}s} + O\left(s^{-\frac{5}{2}}\right) \quad as \quad s \to \infty, \tag{4.11}$$

$$R'(s) \leq C|s|^{-\frac{1}{2}}, \quad s \neq 0. \tag{4.12}$$

Proof. The existence of a function R with $R' > 0$ satisfying (4.6), (4.9), (4.10) was proved in [ABCFFT] using the method of sub-supersolutions. The estimates (4.11), (4.12) follow, and we present their proofs in Appendix A. □

Remark 4.1. *Note that (4.11) is an improvement over the corresponding estimate in* [ABCFFT]*. As we will see it is important that $\frac{5}{2} > 1$.*

We define

$$y_{in}(s) = \epsilon^{\frac{1}{3}} R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right), \quad |s| \leq d_\epsilon. \tag{4.13}$$

Using (4.3), (4.4), (4.13) and Proposition 4.1:

$$|\epsilon^2 y_{in}'' - g_y(x_0, y_{in})| \leq C|\ln \epsilon|^{\frac{5}{2}} \epsilon^{\frac{5}{3}}, \quad \text{for} \quad |s| \leq d_\epsilon, \tag{4.14}$$

and from (4.12):

$$0 < y_{in}'(\pm d_\epsilon) \leq C|\ln \epsilon|^{-\frac{1}{2}} \epsilon^{-\frac{1}{3}}. \tag{4.15}$$

Lemma 4.1. *If $\epsilon > 0$ is sufficiently small then*

$$||y_{in} - y_0||_{L^\infty(-d_\epsilon, d_\epsilon)} \leq C\epsilon^{\frac{1}{3}}, \tag{4.16}$$

$$||y_{in} - y_0||_{L^2(-d_\epsilon, d_\epsilon)} \leq C\epsilon^{\frac{2}{3}}, \tag{4.17}$$

$$||y_0(y_{in} - y_0)||_{L^2(-d_\epsilon, d_\epsilon)} \leq C\epsilon. \tag{4.18}$$

Proof. For $s \geq 0$,

$$\left|\epsilon^{\frac{1}{3}} R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) - y_0(s)\right| \leq \epsilon^{\frac{1}{3}}\left|R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) - \sqrt{\frac{bm}{a}\frac{s}{\epsilon^{\frac{2}{3}}}}\right| + \left|y_0(s) - \sqrt{\frac{bm}{a}s}\right|$$

$$\stackrel{(4.7)}{\leq} \epsilon^{\frac{1}{3}}\left|R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) - \sqrt{\frac{bm}{a}\frac{s}{\epsilon^{\frac{2}{3}}}}\right| + Cs^{\frac{3}{2}}. \tag{4.19}$$

Using (4.11) we get from (4.19) that for $s > 0$:

$$\left|\epsilon^{\frac{1}{3}} R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) - y_0(s)\right| \leq C\epsilon^{\frac{1}{3}}\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right)^{-\frac{5}{2}} + Cs^{\frac{3}{2}} = C\epsilon^2 s^{-\frac{5}{2}} + Cs^{\frac{3}{2}}. \tag{4.20}$$

Proof of (4.16)
For $-d_\epsilon \leq s \leq 0$:

$$|y_{in}(s) - y_0(s)| = \epsilon^{\frac{1}{3}} R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) \stackrel{R\nearrow}{\leq} \epsilon^{\frac{1}{3}} R(0).$$

For $0 \leq s \leq d_\epsilon$ we have from (4.19) and using that $R(t) - \sqrt{\frac{bm}{a}t} \in L^\infty[0,\infty)$:

$$|y_{in}(s) - y_0(s)| = \left|\epsilon^{\frac{1}{3}} R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) - y_0(s)\right| \leq C\epsilon^{\frac{1}{3}} + C|\ln\epsilon|^{\frac{3}{2}}\epsilon \leq C\epsilon^{\frac{1}{3}}.$$

Relation (4.16) now follows.
Proof of (4.17)
We have

$$\begin{aligned}||y_{in} - y_0||^2_{L^2(-d_\epsilon,0)} &= \int_{-|\ln\epsilon|\epsilon^{\frac{2}{3}}}^{0} \epsilon^{\frac{2}{3}} R^2\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) ds = \\ &= \epsilon^{\frac{4}{3}}\int_{-|\ln\epsilon|}^{0} R^2(t)dt \stackrel{(4.10)}{\leq} C\epsilon^{\frac{4}{3}}\int_{-\infty}^{0} e^{2t}dt \leq C\epsilon^{\frac{4}{3}},\end{aligned}$$

$$\begin{aligned}||y_{in} - y_0||^2_{L^2(0,d_\epsilon)} &= \int_0^{|\ln\epsilon|\epsilon^{\frac{2}{3}}} \left(\epsilon^{\frac{1}{3}} R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) - y_0(s)\right)^2 ds = \\ &= \int_0^{\epsilon^{\frac{2}{3}}} \left(\epsilon^{\frac{1}{3}} R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) - y_0(s)\right)^2 ds + \int_{\epsilon^{\frac{2}{3}}}^{|\ln\epsilon|\epsilon^{\frac{2}{3}}} \left(\epsilon^{\frac{1}{3}} R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) - y_0(s)\right)^2 ds \leq \\ &\stackrel{(4.20)}{\leq} 2\int_0^{\epsilon^{\frac{2}{3}}} \left(\epsilon^{\frac{2}{3}} R^2\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) + y_0^2(s)\right) ds + C\int_{\epsilon^{\frac{2}{3}}}^{|\ln\epsilon|\epsilon^{\frac{2}{3}}} (\epsilon^4 s^{-5} + s^3) ds \stackrel{(3.14)}{\leq} C\epsilon^{\frac{4}{3}}.\end{aligned}$$

Relation (4.17) follows.
Proof of (4.18)
Relation (4.18) is proved similarly to (4.17) using that $y_0(s) = 0,\ s \leq 0$, and (3.14). □

<u>The outer approximations $y_{out}^{\epsilon,-}$, $y_{out}^{\epsilon,+}$</u>

We seek

$$y^+_{out}(s) = y_0(s) + \sigma_+(s), \quad s \geq d_\epsilon, \tag{4.21}$$

such that

$$y_{out}^+(d_\epsilon) = y_{in}(d_\epsilon), \qquad y_{out}^+(\infty) = \sqrt{2}. \tag{4.22}$$

Since $g_y(x_0, y_0) = 0$, we have for $s \geq d_\epsilon$:

$$\begin{aligned} -\epsilon^2 {y_{out}^+}'' + g_y(x_0, y_{out}^+) &= -\epsilon^2\sigma_+'' + g_{yy}(x_0,y_0)\sigma_+ - \epsilon^2 y_0'' + \\ &\quad + g_y(x_0, y_0+\sigma_+) - g_y(x_0,y_0) - g_{yy}(x_0,y_0)\sigma_+ \end{aligned}$$

or from (2.5), (2.6)

$$-\epsilon^2 {y_{out}^+}'' + g_y(x_0, y_{out}^+) = -\epsilon^2\sigma_+'' + g_{yy}(x_0,y_0)\sigma_+ - \epsilon^2 y_0'' + 3\sigma_+^3 + 9y_0\sigma_+^2, \qquad s \geq d_\epsilon. \tag{4.23}$$

We choose σ_+ such that

$$-\epsilon^2\sigma_+'' + g_{yy}(x_0,y_0)\sigma_+ = \epsilon^2 y_0'', \qquad s \geq d_\epsilon, \tag{4.24}$$

and (from (4.22))

$$\sigma_+(d_\epsilon) = \epsilon^{\frac{1}{3}} R(|\ln\epsilon|) - y_0(d_\epsilon), \quad \sigma_+(\infty) = 0. \tag{4.25}$$

Analogously we seek

$$y_{out}^-(s) = y_0(s) + \sigma_-(s) = \sigma_-(s), \qquad s \leq -d_\epsilon, \tag{4.26}$$

such that

$$y_{out}^-(-\infty) = 0, \qquad y_{out}^-(-d_\epsilon) = y_{in}(-d_\epsilon). \tag{4.27}$$

We have

$$-\epsilon^2 {y_{out}^-}'' + g_y(x_0, y_{out}^-) = -\epsilon^2\sigma_-'' + g_{yy}(x_0,y_0)\sigma_- + 3\sigma_-^3, \qquad s \leq -d_\epsilon. \tag{4.28}$$

We choose σ_- such that

$$-\epsilon^2\sigma_-'' + g_{yy}(x_0,y_0)\sigma_- = 0, \qquad s \leq -d_\epsilon, \tag{4.29}$$

and

$$\sigma_-(-\infty) = 0, \qquad \sigma_-(-d_\epsilon) = \epsilon^{\frac{1}{3}} R(-|\ln\epsilon|). \tag{4.30}$$

We note that there exist unique $\sigma_+ \in H^2(d_\epsilon, \infty)$, $\sigma_- \in H^2(-\infty, -d_\epsilon)$ solutions of (4.24), (4.25) and (4.29), (4.30) respectively. This follows from $(4.31)_{(i)}$ below and $y_0'' \in L^2(d_\epsilon, \infty)$ (cf. (3.13)).

In the sequel we will make use of the following relations which follow easily from (2.6), (3.11), (3.14), (3.15), and we cite them here for future reference (see Figure 3):

$$\begin{cases} g_{yy}(x_0(s), y_0(s)) \text{ is strictly decreasing/increasing for } s<0/s>0,\ g_{yy}(x_0(0), y_0(0)) = 0, \\ g_{yy}(x_0(s), y_0(s)) \geq c|s|,\ |s| \leq 1, \text{ and } g_{yy}(x_0(s), y_0(s)) \leq C|s|,\ s \in \mathbb{R}. \end{cases} \tag{4.31}$$

Remark 4.2. *Only mild use of the monotonicity of $g_{yy}(x_0(\cdot), y_0(\cdot))$ and y_0 will be made in this paper.*

Lemma 4.2. *If $\epsilon > 0$ is sufficiently small, then*

$$||\sigma_-||_{L^\infty(-\infty,-d_\epsilon)}, \quad ||\sigma_+||_{L^\infty(d_\epsilon,\infty)} \leq C|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}}, \tag{4.32}$$

$$||y_0\sigma_+||_{L^\infty(d_\epsilon,\infty)} \leq C|\ln\epsilon|^{-2}\epsilon^{\frac{2}{3}}, \tag{4.33}$$

$$|\sigma_\pm'(\pm d_\epsilon)| \leq C|\ln\epsilon|^{-\frac{3}{2}}\epsilon^{-\frac{1}{3}}, \tag{4.34}$$

$$||\sigma_-||_{L^2(-\infty,-d_\epsilon)}, \quad ||\sigma_+||_{L^2(d_\epsilon,\infty)} \le C|\ln\epsilon|^{-2}\epsilon^{\frac{2}{3}}, \tag{4.35}$$

$$||y_0\sigma_+||_{L^2(d_\epsilon,\infty)} \le C|\ln\epsilon|^{-\frac{3}{2}}\epsilon. \tag{4.36}$$

Proof. Proof of (4.32)
We have from (4.25):

$$|\sigma_+(d_\epsilon)| = \left|\epsilon^{\frac{1}{3}}R(|\ln\epsilon|) - y_0(|\ln\epsilon|\epsilon^{\frac{2}{3}})\right| \overset{(4.20)}{\le}$$

$$\le C|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}} + C|\ln\epsilon|^{\frac{3}{2}}\epsilon \le C|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}} \tag{4.37}$$

if ϵ is sufficiently small.
Let $\max_{s\ge d_\epsilon}\sigma_+(s) = \sigma_+(s_0)$ (recall $\sigma_+(\infty)=0$). If $s_0 = d_\epsilon$, then via (4.37),

$$\sigma_+(s) \le C|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}}, \quad s \ge d_\epsilon.$$

If $s_0 > d_\epsilon$, then $\sigma_+'(s_0)=0$ and $\sigma_+''(s_0)\le 0$. Substituting $s=s_0$ in (4.24):

$$g_{yy}(x_0(s_0),y_0(s_0))\sigma_+(s_0) \le \epsilon^2 y_0''(s_0). \tag{4.38}$$

Note that from (4.31) we get

$$g_{yy}(x_0(s),y_0(s)) \ge c|\ln\epsilon|\epsilon^{\frac{2}{3}}, \quad |s|\ge d_\epsilon \tag{4.39}$$

and from (3.14):

$$|y_0''(s)| \le Cs^{-\frac{3}{2}} \le C|\ln\epsilon|^{-\frac{3}{2}}\epsilon^{-1}, \quad s\ge d_\epsilon. \tag{4.40}$$

Thus from (4.38):

$$\sigma_+(s_0) \le \frac{\epsilon^2 y_0''(s_0)}{g_{yy}(x_0(s_0),y_0(s_0))} \overset{(4.39),(4.40)}{\le} C\frac{|\ln\epsilon|^{-\frac{3}{2}}\epsilon}{|\ln\epsilon|\epsilon^{\frac{2}{3}}} = C|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}}$$

i.e.

$$\sigma_+(s) \le C|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}}, \quad s\ge d_\epsilon.$$

By applying the above procedure to $-\sigma_+$, we obtain (4.32) for σ_+. The proof of (4.32) for σ_- is identical.
Proof of (4.33)
We set $\Sigma(s) = y_0(s)\sigma_+(s),\ s\ge d_\epsilon$, then

$$|\Sigma(d_\epsilon)| \overset{(3.14),(4.32)}{\le} C|\ln\epsilon|^{\frac{1}{2}}\epsilon^{\frac{1}{3}}|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}} = C|\ln\epsilon|^{-2}\epsilon^{\frac{2}{3}}. \tag{4.41}$$

From (4.24):

$$-\epsilon^2\Sigma'' + g_{yy}(x_0,y_0)\Sigma = -\epsilon^2 y_0''\sigma_+ - 2\epsilon^2 y_0'\sigma_+' + \epsilon^2 y_0 y_0'', \qquad s\ge d_\epsilon. \tag{4.42}$$

Let $\max_{s\ge d_\epsilon}\Sigma(s) = \Sigma(s_0)$ (recall $\Sigma(\infty)=0$). If $s_0 = d_\epsilon$, then (4.41) gives

$$\Sigma(s) \le C|\ln\epsilon|^{-2}\epsilon^{\frac{2}{3}}, \quad s\ge d_\epsilon.$$

If $s_0 > d_\epsilon$, then $\Sigma'(s_0)=0$ and $\Sigma''(s_0)\le 0$. Hence

$$\sigma_+'(s_0) = -\frac{y_0'(s_0)}{y_0(s_0)}\sigma_+(s_0). \tag{4.43}$$

Note that via (3.11), (3.15),

$$y_0(s) \ge c|\ln\epsilon|^{\frac{1}{2}}\epsilon^{\frac{1}{3}}, \quad s\ge d_\epsilon \tag{4.44}$$

and from (3.14), (4.32), (4.44):
(4.45)
$$|\sigma_+'(s_0)| \leq C\frac{s_0^{-\frac{1}{2}}}{|\ln\epsilon|^{\frac{1}{2}}\epsilon^{\frac{1}{3}}}|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}} \leq C\frac{|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{-\frac{1}{3}}|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}}}{|\ln\epsilon|^{\frac{1}{2}}\epsilon^{\frac{1}{3}}} = C|\ln\epsilon|^{-\frac{7}{2}}\epsilon^{-\frac{1}{3}}.$$

Substituting $s = s_0$ in (4.42) and using that $\Sigma''(s_0) \leq 0$:

$$g_{yy}(x_0(s_0), y_0(s_0))\Sigma(s_0) \leq |-\epsilon^2 y_0''(s_0)\sigma_+(s_0) - 2\epsilon^2 y_0'(s_0)\sigma_+'(s_0) + \epsilon^2 y_0(s_0) y_0''(s_0)| \leq$$
$$\overset{(3.14),(4.32),(4.45)}{\leq} C\epsilon^2 s_0^{-\frac{3}{2}}|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}} + C\epsilon^2 s_0^{-\frac{1}{2}}|\ln\epsilon|^{-\frac{7}{2}}\epsilon^{-\frac{1}{3}} + C\epsilon^2 s_0^{\frac{1}{2}} s_0^{-\frac{3}{2}} \leq$$
$$\leq C\epsilon^2|\ln\epsilon|^{-\frac{3}{2}}\epsilon^{-1}|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}} + C\epsilon^2|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{-\frac{1}{3}}|\ln\epsilon|^{-\frac{7}{2}}\epsilon^{-\frac{1}{3}} + C\epsilon^2|\ln\epsilon|^{-1}\epsilon^{-\frac{2}{3}} \leq C|\ln\epsilon|^{-1}\epsilon^{\frac{4}{3}}.$$

Now via (4.39), we have $\Sigma(s_0) \leq C|\ln\epsilon|^{-2}\epsilon^{\frac{2}{3}}$, i.e. $\Sigma(s) \leq C|\ln\epsilon|^{-2}\epsilon^{\frac{2}{3}}$, $s \geq d_\epsilon$. Relation (4.33) follows from the same procedure applied to $-\Sigma$.

Proof of (4.34)

Multiplying (4.24) with $\epsilon^{-2}\left(s-(d_\epsilon+\epsilon^{\frac{2}{3}})\right)$, gives for $s \geq d_\epsilon$:

$$-\left(s-(d_\epsilon+\epsilon^{\frac{2}{3}})\right)\sigma_+'' + \epsilon^{-2}\left(s-(d_\epsilon+\epsilon^{\frac{2}{3}})\right)g_{yy}(x_0,y_0)\sigma_+ = \left(s-(d_\epsilon+\epsilon^{\frac{2}{3}})\right)y_0''.$$

Integrating over $\left[d_\epsilon, d_\epsilon+\epsilon^{\frac{2}{3}}\right]$ and noting that $\left(s-(d_\epsilon+\epsilon^{\frac{2}{3}})\right)\sigma_+'' = \left[\left(s-(d_\epsilon+\epsilon^{\frac{2}{3}})\right)\sigma_+' - \sigma_+\right]'$, gives

$$\sigma_+(d_\epsilon+\epsilon^{\frac{2}{3}}) - \epsilon^{\frac{2}{3}}\sigma_+'(d_\epsilon) - \sigma_+(d_\epsilon) +$$
$$+\epsilon^{-2}\int_{d_\epsilon}^{d_\epsilon+\epsilon^{\frac{2}{3}}}\left(s-(d_\epsilon+\epsilon^{\frac{2}{3}})\right)g_{yy}(x_0,y_0)\sigma_+ ds = \int_{d_\epsilon}^{d_\epsilon+\epsilon^{\frac{2}{3}}}\left(s-(d_\epsilon+\epsilon^{\frac{2}{3}})\right)y_0'' ds.$$

Using (4.32), (4.31), (4.40) we have

$$\epsilon^{\frac{2}{3}}|\sigma_+'(d_\epsilon)| \leq C|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}} + C\epsilon^{-2}\epsilon^{\frac{2}{3}}\epsilon^{\frac{2}{3}}(|\ln\epsilon|+1)\epsilon^{\frac{2}{3}}|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}} + C\epsilon^{\frac{2}{3}}\epsilon^{\frac{2}{3}}|\ln\epsilon|^{-\frac{3}{2}}\epsilon^{-1} \leq C|\ln\epsilon|^{-\frac{3}{2}}\epsilon^{\frac{1}{3}}$$

i.e we obtain (4.34) for σ_+. The proof of (4.34) for σ_- is analogous. We note that in order to estimate the second term of the right hand side of the above relation it was important that $\frac{5}{2} > 1$ (cf. Remark 4.1).

Proof of (4.35)

Multiplying (4.24) with σ_+ and integrating over $[d_\epsilon, \infty)$, gives

$$-\epsilon^2\int_{d_\epsilon}^{\infty}\sigma_+''\sigma_+ ds + \int_{d_\epsilon}^{\infty}g_{yy}(x_0,y_0)\sigma_+^2 ds = \epsilon^2\int_{d_\epsilon}^{\infty}y_0''\sigma_+ ds.$$

Integrating by parts (recall $\sigma_+ \in H^2(d_\epsilon,\infty)$):

$$\epsilon^2\sigma_+'(d_\epsilon)\sigma_+(d_\epsilon) + \epsilon^2\int_{d_\epsilon}^{\infty}{\sigma_+'}^2 ds + \int_{d_\epsilon}^{\infty}g_{yy}(x_0,y_0)\sigma_+^2 ds = \epsilon^2\int_{d_\epsilon}^{\infty}y_0''\sigma_+ ds$$

and by using (4.32), (4.34), (3.14),

$$\int_{d_\epsilon}^{\infty}g_{yy}(x_0,y_0)\sigma_+^2 ds \leq C\epsilon^2|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}}\int_{|\ln\epsilon|\epsilon^{\frac{2}{3}}}^{\infty}s^{-\frac{3}{2}}ds + C\epsilon^2|\ln\epsilon|^{-\frac{3}{2}}\epsilon^{-\frac{1}{3}}|\ln\epsilon|^{-\frac{5}{2}}\epsilon^{\frac{1}{3}},$$

or

$$\int_{d_\epsilon}^{\infty}g_{yy}(x_0,y_0)\sigma_+^2 ds \leq C|\ln\epsilon|^{-3}\epsilon^2. \tag{4.46}$$

Thus via (4.39)

$$\int_{d_\epsilon}^{\infty}\sigma_+^2 ds \leq C|\ln\epsilon|^{-4}\epsilon^{\frac{4}{3}},$$

which proves (4.35) for σ_+. The proof of (4.35) for σ_- is analogous.

Proof of (4.36)

Relation (4.36) is a consequence of (4.46) and the observation (cf. (3.14), (4.31))

$$g_{yy}(x_0(s), y_0(s)) \geq c y_0^2(s), \qquad s \in \mathbb{R}. \tag{4.47}$$

The proof of the Lemma is complete. □

From (4.23), (4.28) via (4.24), (4.29), (4.32), (4.33), (4.35) and the definition

$$y_{out}(s) := \begin{cases} y_{out}^-(s), & s \leq -d_\epsilon \\ y_{out}^+(s), & s \geq d_\epsilon \end{cases}$$

we obtain

$$|| - \epsilon^2 y_{out}'' + g_y(x_0, y_{out})||_{L^\infty(|s| \geq d_\epsilon)} \leq C|\ln \epsilon|^{-\frac{9}{2}} \epsilon, \tag{4.48}$$

$$|| - \epsilon^2 y_{out}'' + g_y(x_0, y_{out})||_{L^2(|s| \geq d_\epsilon)} \leq C|\ln \epsilon|^{-4} \epsilon^{\frac{4}{3}}. \tag{4.49}$$

From (4.21), (4.26), (3.14), (4.34):

$$|y_{out}'(\pm d_\epsilon)| \leq C|\ln \epsilon|^{-\frac{1}{2}} \epsilon^{-\frac{1}{3}} \tag{4.50}$$

and from (4.32), (4.35), (4.36):

$$||y_{out} - y_0||_{L^\infty(|s| \geq d_\epsilon)} \leq C|\ln \epsilon|^{-\frac{5}{2}} \epsilon^{\frac{1}{3}}, \tag{4.51}$$

$$||y_{out} - y_0||_{L^2(|s| \geq d_\epsilon)} \leq C|\ln \epsilon|^{-2} \epsilon^{\frac{2}{3}}, \tag{4.52}$$

$$||y_0(y_{out} - y_0)||_{L^2(|s| \geq d_\epsilon)} \leq C|\ln \epsilon|^{-\frac{3}{2}} \epsilon. \tag{4.53}$$

<u>The function y_{ap}^ϵ</u>

We define $y_{ap}^\epsilon \in C(\mathbb{R}) \cap C^\infty(\mathbb{R} - \{\pm d_\epsilon\})$ by

$$y_{ap}^\epsilon = \begin{cases} y_{out}, & |s| \geq d_\epsilon \\ y_{in}, & |s| \leq d_\epsilon \end{cases} \tag{4.54}$$

(cf. (4.1)).

It follows from (4.22), (4.27) that

$$y_{ap}(-\infty) = 0, \qquad y_{ap}(\infty) = \sqrt{2} \tag{4.55}$$

and from (4.14), (4.48), (4.49) that

$$| - \epsilon^2 y_{ap}'' + g_y(x_0, y_{ap})| \leq C|\ln \epsilon|^{-\frac{9}{2}} \epsilon, \qquad s \neq \pm d_\epsilon, \tag{4.56}$$

$$\begin{aligned} &|| - \epsilon^2 y_{ap}'' + g_y(x_0, y_{ap})||_{L^2(-\infty, -d_\epsilon)} \\ &|| - \epsilon^2 y_{ap}'' + g_y(x_0, y_{ap})||_{L^2(-d_\epsilon, d_\epsilon)} \quad \leq \quad C|\ln \epsilon|^{-4} \epsilon^{\frac{4}{3}} \\ &|| - \epsilon^2 y_{ap}'' + g_y(x_0, y_{ap})||_{L^2(d_\epsilon, \infty)} \end{aligned} \tag{4.57}$$

Furthermore (4.15), (4.50) imply

$$|y_{ap}'(\pm d_\epsilon^\pm)| \leq C|\ln \epsilon|^{-\frac{1}{2}} \epsilon^{-\frac{1}{3}} \tag{4.58}$$

and (4.16), (4.17), (4.18), (4.51), (4.52), (4.53):

$$||y_{ap} - y_0||_{L^\infty(\mathbb{R})} \le C\epsilon^{\frac{1}{3}}, \tag{4.59}$$

$$||y_{ap} - y_0||_{L^2(\mathbb{R})} \le C\epsilon^{\frac{2}{3}}, \tag{4.60}$$

$$||y_0(y_{ap} - y_0)||_{L^2(\mathbb{R})} \le C\epsilon. \tag{4.61}$$

The function $\hat{\rho}^\epsilon$

As we will see (cf. (5.29)), there exists a large $D > 0$ (independent of ϵ) such that

$$g_{yy}(x_0(s), y_{ap}(s)) \ge c\epsilon^{\frac{2}{3}} \quad \text{for} \quad |s| \ge D\epsilon^{\frac{2}{3}}.$$

We define a function $\tilde{g}_{yy}(\cdot) \in C(\mathbb{R})$ by

$$\tilde{g}_{yy}(s) = \begin{cases} g_{yy}(x_0(s), y_{ap}(s)), & |s| \ge D\epsilon^{\frac{2}{3}} \\ \text{linear}, & |s| \le D\epsilon^{\frac{2}{3}} \end{cases}$$

Then it is easy to see that

$$c\epsilon^{\frac{2}{3}} \le \tilde{g}_{yy}(s) \le C, \qquad s \in \mathbb{R} \tag{4.62}$$

and

$$||g_{yy}(x_0(s), y_{ap}(s)) - \tilde{g}_{yy}(s)||_{L^\infty(\mathbb{R})} \le C\epsilon^{\frac{2}{3}}. \tag{4.63}$$

We also define two functions $\rho_\pm \in H^1(\mathbb{R})$ from the relations

$$-\epsilon^2\rho_\pm'' + \tilde{g}_{yy}(s)\rho_\pm = 0, \quad s \ne \pm d_\epsilon \tag{4.64}$$

and

$$\rho_-'(-d_\epsilon^-) - \rho_-'(-d_\epsilon^+) = 1, \tag{4.65}$$

$$\rho_+'(d_\epsilon^-) - \rho_+'(d_\epsilon^+) = 1. \tag{4.66}$$

The existence and uniqueness of $\rho_\pm > 0$ follows from (4.62). Using (4.62), (4.64), (4.65), (4.66) and the maximum principle we have

$$0 < \rho_\pm(s) \le \frac{1}{2}\frac{\epsilon^{\frac{2}{3}}}{\sqrt{c}} e^{-\sqrt{c}\frac{|s \mp d_\epsilon|}{\epsilon^{\frac{2}{3}}}}, \quad s \in \mathbb{R} \tag{4.67}$$

(c as in (4.62)).
From (4.67), (3.14) one can check that the following hold:

$$||\rho_\pm||_{L^\infty(\mathbb{R})} \le C\epsilon^{\frac{2}{3}}, \tag{4.68}$$

$$||\rho_\pm||_{L^2(\mathbb{R})} \le C\epsilon, \tag{4.69}$$

$$||y_0\rho_\pm||_{L^\infty(\mathbb{R})} \le C|\ln\epsilon|^{\frac{1}{2}}\epsilon, \tag{4.70}$$

$$||y_0\rho_\pm||_{L^2(\mathbb{R})} \le C|\ln\epsilon|^{\frac{1}{2}}\epsilon^{\frac{4}{3}}. \tag{4.71}$$

Let

$$\hat{\rho}(s) = \left(y_{ap}'(-d_\epsilon^+) - y_{ap}'(-d_\epsilon^-)\right)\rho_-(s) + \left(y_{ap}'(d_\epsilon^+) - y_{ap}'(d_\epsilon^-)\right)\rho_+(s). \tag{4.72}$$

Then (4.68), (4.69), (4.70), (4.71) and (4.58) imply:

$$||\hat{\rho}||_{L^\infty(\mathbb{R})} \leq C|\ln \epsilon|^{-\frac{1}{2}}\epsilon^{\frac{1}{3}}, \tag{4.73}$$

$$||\hat{\rho}||_{L^2(\mathbb{R})} \leq C|\ln \epsilon|^{-\frac{1}{2}}\epsilon^{\frac{2}{3}}, \tag{4.74}$$

$$||y_0\hat{\rho}||_{L^\infty(\mathbb{R})} \leq C\epsilon^{\frac{2}{3}}, \tag{4.75}$$

$$||y_0\hat{\rho}||_{L^2(\mathbb{R})} \leq C\epsilon. \tag{4.76}$$

Also via (4.73), (4.75), (4.59):

$$||y_{ap}\hat{\rho}||_{L^\infty(\mathbb{R})} \leq C\epsilon^{\frac{2}{3}}, \tag{4.77}$$

$$||y_{ap}\hat{\rho}||_{L^2(\mathbb{R})} \leq C\epsilon. \tag{4.78}$$

<u>The function Y^ϵ_{ap}</u>

Let $Y^\epsilon_{ap} := y_{ap}+\hat{\rho}$ (cf. (4.2)). It follows from (4.65), (4.66), (4.72) that $Y_{ap} \in C^1(\mathbb{R})$. We note that

$$Y_{ap}(-\infty) = 0, \qquad Y_{ap}(\infty) = \sqrt{2}, \tag{4.79}$$

$Y'_{ap} \in H^1(\mathbb{R})$ and Y''_{ap} has exactly two points of finite discontinuity at $s = \pm d_\epsilon$.

Proposition 4.2. *If $\epsilon > 0$ is sufficiently small, then the following hold:*

$$||\epsilon^2 {Y_{ap}}'' - g_y(x_0, Y_{ap})||_{L^\infty(\mathbb{R})} \leq C|\ln \epsilon|^{-\frac{1}{2}}\epsilon, \tag{4.80}$$

$$||\epsilon^2 {Y_{ap}}'' - g_y(x_0, Y_{ap})||_{L^2(\mathbb{R})} \leq C|\ln \epsilon|^{-\frac{1}{2}}\epsilon^{\frac{4}{3}}, \tag{4.81}$$

$$||Y_{ap} - y_0||_{L^\infty(\mathbb{R})} \leq C\epsilon^{\frac{1}{3}}, \tag{4.82}$$

$$||Y_{ap} - y_0||_{L^2(\mathbb{R})} \leq C\epsilon^{\frac{2}{3}}, \tag{4.83}$$

$$||y_0(Y_{ap} - y_0)||_{L^2(\mathbb{R})} \leq C\epsilon. \tag{4.84}$$

Proof. For $s \neq \pm d_\epsilon$:

$$-\epsilon^2 {Y_{ap}}'' + g_y(x_0, Y_{ap}) = -\epsilon^2 {y_{ap}}'' - \epsilon^2\hat{\rho}'' + g_y(x_0, y_{ap}+\hat{\rho}) =$$

$$= -\epsilon^2 {y_{ap}}'' + g_y(x_0, y_{ap}) + g_y(x_0, y_{ap}+\hat{\rho}) - g_y(x_0, y_{ap}) - g_{yy}(x_0, y_{ap})\hat{\rho} + \tag{4.85}$$

$$+(g_{yy}(x_0, y_{ap}) - \tilde{g}_{yy}(s))\hat{\rho} =$$

$$= -\epsilon^2 {y_{ap}}'' + g_y(x_0, y_{ap}) + 3\hat{\rho}^3 + 9y_{ap}\hat{\rho}^2 + (g_{yy}(x_0, y_{ap}) - \tilde{g}_{yy}(s))\hat{\rho}$$

Relation (4.80) follows from (4.85) via (4.56), (4.73), (4.77), (4.63). Relation (4.81) follows from (4.85) via (4.57), (4.73), (4.74), (4.77), (4.63). Relation (4.82) follows from (4.59), (4.73). Relation (4.83) follows from (4.60), (4.74). Relation (4.84) follows from (4.61), (4.76). The proof of the Proposition is complete. □

5. The linear operator L_ϵ

In this section we will study the linear operator

$$L_\epsilon y = -\epsilon^2 y'' + g_{yy}(x_0, \tilde{Y})y$$

where $\tilde{Y}(s) = y_{ap}(s) + \tilde{\rho}(s),\ s \in \mathbb{R}$ and $\tilde{\rho} \in C(\mathbb{R})$ such that

$$||\tilde{\rho}||_{L^\infty(\mathbb{R})} \le C|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{1}{3}}. \tag{5.1}$$

Remark 5.1. *The results of this section also hold for* $||\tilde{\rho}||_{L^\infty(\mathbb{R})} \le C|\ln\epsilon|^{-\alpha}\epsilon^{\frac{1}{3}}$, $\alpha > 0$.

Lemma 5.1. *There exists a $D > 0$ that depends only on the function R (thus it is independent of ϵ) such that if $\epsilon > 0$ is sufficiently small, we have*

$$g_{yy}(x_0(s), \tilde{Y}(s)) \ge c|s| \quad \textit{for} \quad D\epsilon^{\frac{2}{3}} \le |s| \le 1, \tag{5.2}$$

$$g_{yy}(x_0(s), \tilde{Y}(s)) \ge c \qquad \textit{for} \quad |s| \ge 1. \tag{5.3}$$

Proof. We will prove (5.2) for $D\epsilon^{\frac{2}{3}} \le s \le 1$ and leave the rest to the reader.

Relation (4.11) implies that there exists a large $D > 0$ such that

$$R(s) \ge \frac{1}{\sqrt{2}}\sqrt{\frac{bm}{a}s}, \qquad s \ge D. \tag{5.4}$$

Note that from (2.10):

$$g_{yy}(x_0(s), \tilde{Y}(s)) = 9\tilde{Y}^2(s) - 3y_0^2(s), \quad s \ge 0. \tag{5.5}$$

For $D\epsilon^{\frac{2}{3}} \le s \le |\ln\epsilon|\epsilon^{\frac{2}{3}}$ we have $\tilde{Y}(s) = \epsilon^{\frac{1}{3}} R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) + \tilde{\rho}(s)$ and thus from (5.5):

$$\begin{aligned} g_{yy}(x_0(s), \tilde{Y}(s)) &= 9\epsilon^{\frac{2}{3}} R^2\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) + 9\tilde{\rho}^2(s) + 18\epsilon^{\frac{1}{3}} R\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right)\tilde{\rho}(s) - 3y_0^2(s) \\ &\ge 8\epsilon^{\frac{2}{3}} R^2\left(\frac{s}{\epsilon^{\frac{2}{3}}}\right) - 72\tilde{\rho}^2(s) - 3y_0^2(s). \end{aligned}$$

Since $\frac{s}{\epsilon^{\frac{2}{3}}} \ge D$, relation (5.4) yields

$$\begin{aligned} g_{yy}(x_0(s), \tilde{Y}(s)) &\ge 4\frac{bm}{a}s - 72\tilde{\rho}^2(s) - 3y_0^2(s) \\ &= \frac{bm}{a}s - 72\tilde{\rho}^2 + 3\left(\frac{bm}{a}s - y_0^2(s)\right) \\ &\overset{(5.1),(2.10)}{\ge} \frac{bm}{a}s - C|\ln\epsilon|^{-1}\epsilon^{\frac{2}{3}} - Cs^2 \\ &\ge \frac{bm}{a}s - C|\ln\epsilon|^{-1}\frac{s}{D} - C|\ln\epsilon|\epsilon^{\frac{2}{3}}s \ge cs \end{aligned}$$

provided $\epsilon > 0$ is sufficiently small.

For $|\ln\epsilon|\epsilon^{\frac{2}{3}} \le s$ we have that $\tilde{Y}(s) = y_0(s) + \sigma_+(s) + \tilde{\rho}(s)$ and by working as before:

$$g_{yy}(x_0(s), \tilde{Y}(s)) \ge 5y_0^2(s) - 72(\sigma_+(s) + \tilde{\rho}(s))^2, \quad s \ge |\ln\epsilon|\epsilon^{\frac{2}{3}}. \tag{5.6}$$

Thus for $|\ln\epsilon|\epsilon^{\frac{2}{3}} \leq s \leq 1$ we obtain via (3.15), (4.32), (5.1):

$$g_{yy}(x_0(s), \tilde{Y}(s)) \geq cs - C|\ln\epsilon|^{-1}\epsilon^{\frac{2}{3}} \geq cs - C|\ln\epsilon|^{-1}\frac{s}{|\ln\epsilon|} \geq cs$$

provided $\epsilon > 0$ is sufficiently small.
The result follows by combining the two above cases. □

The following Proposition will be proved in Appendix B.

Proposition 5.1. *Suppose that* $y \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and* $\mu \leq 0$ *satisfy*

$$-y'' + 2(3aR^2(s) - bms)y = \mu y, \quad s \in \mathbb{R}. \tag{5.7}$$

Then $y \equiv 0$.

Proposition 5.2. *Let* $y \in H^2(\mathbb{R})$, $f \in L^\infty(\mathbb{R})$ *with isolated points of discontinuity and*

$$-\epsilon^2 y'' + g_{yy}(x_0(s), \tilde{Y}(s))y = f, \quad s \in \mathbb{R}.$$

Then if $\epsilon > 0$ *is sufficiently small (independent of* y, f*), we have*

$$||y||_{L^\infty(\mathbb{R})} \leq C\epsilon^{-\frac{2}{3}}||f||_{L^\infty(\mathbb{R})}$$

($C > 0$ *independent of* y, f, ϵ*).*

Proof. We will use a "blow up" argument. Suppose that there exist $\epsilon_n > 0$, $y_n \in H^2(\mathbb{R})$ and $f_n \in L^\infty(\mathbb{R})$ with isolated points of discontinuity, such that

$$-\epsilon_n^2 y_n'' + g_{yy}(x_0(s), \tilde{Y}(s))y_n = f_n, \quad s \in \mathbb{R}, \tag{5.8}$$

$$\epsilon_n \to 0\ , \quad ||y_n||_{L^\infty(\mathbb{R})} = 1\ , \quad \epsilon_n^{-\frac{2}{3}}||f_n||_{L^\infty(\mathbb{R})} \to 0\ , \quad n \to \infty. \tag{5.9}$$

Let $\tilde{y}_n(s) = y_n(\epsilon_n^{\frac{2}{3}} s)$, $\tilde{f}_n(s) = f_n(\epsilon_n^{\frac{2}{3}} s)$, $s \in \mathbb{R}$, $n \geq 1$, then

$$-\tilde{y}_n'' + \epsilon_n^{-\frac{2}{3}} g_{yy}(x_0(\epsilon_n^{\frac{2}{3}} s), \tilde{Y}(\epsilon_n^{\frac{2}{3}} s))\tilde{y}_n = \epsilon_n^{-\frac{2}{3}}\tilde{f}_n(s), \quad s \in \mathbb{R}, \tag{5.10}$$

$$||\tilde{y}_n||_{L^\infty(\mathbb{R})} = 1\ , \quad \epsilon_n^{-\frac{2}{3}}||\tilde{f}_n||_{L^\infty(\mathbb{R})} \to 0\ , \quad n \to \infty. \tag{5.11}$$

Note that

$$\epsilon^{-\frac{2}{3}} g_{yy}(x_0(\epsilon^{\frac{2}{3}} s), \tilde{Y}(\epsilon^{\frac{2}{3}} s)) \overset{C_{loc}(\mathbb{R})}{\to} 2(3aR^2(s) - bms) \quad \text{as} \quad \epsilon \to 0. \tag{5.12}$$

Indeed, in $[-L, L]$ with $L > 0$ (independent of $\epsilon > 0$), we have for small $\epsilon > 0$ ($|\ln\epsilon| > L$):

$$\tilde{Y}(\epsilon^{\frac{2}{3}} s) = \epsilon^{\frac{1}{3}} R(s) + \tilde{\rho}(\epsilon^{\frac{2}{3}} s)$$

and from (2.10) we get that for $|s| \leq L$:

$$\begin{aligned}\epsilon^{-\frac{2}{3}} g_{yy}(x_0(\epsilon^{\frac{2}{3}} s), \tilde{Y}(\epsilon^{\frac{2}{3}} s)) \quad = \quad & 6aR^2(s) + 12a\epsilon^{-\frac{1}{3}} R(s)\tilde{\rho}(\epsilon^{\frac{2}{3}} s) + 6a\epsilon^{-\frac{2}{3}}\tilde{\rho}^2(\epsilon^{\frac{2}{3}} s) - \\ & -2b\epsilon^{-\frac{2}{3}}(x_0(\epsilon^{\frac{2}{3}} s) - x_0(0)) + 2\epsilon^{-\frac{2}{3}}(x_0(\epsilon^{\frac{2}{3}} s) - x_0(0))^2\end{aligned}$$

Relation (5.12) follows from the above relation, (5.1) and $m = x_0'(0)$.

From (5.10), (5.11), (5.12) and the usual diagonal argument we obtain (passing to a subsequence) that $\tilde{y}_n \overset{H^2_{loc}(\mathbb{R})}{\to} \bar{y}$, $n \to \infty$ with

$$-\bar{y}'' + 2(3aR^2(s) - bms)\bar{y} = 0, \quad s \in \mathbb{R} \tag{5.13}$$

and

$$||\bar{y}||_{L^\infty(\mathbb{R})} \leq 1. \tag{5.14}$$

To finish, we show that $\bar{y}$ is not identically zero. Indeed, (5.10) and the regularity assumption on f_n yield that each $\tilde{y}_n \in C^2(\mathbb{R} - \{\text{isolated points}\}) \cap C^1(\mathbb{R})$. Thus from (5.11) we may assume without loss of generality that there exists a sequence $\{s_n\}$, $n \geq 1$ such that

$$\tilde{y}_n(s_n) \geq \frac{1}{2} \quad , \quad \tilde{y}_n''(s_n) \leq 0 \tag{5.15}$$

(there exists a sequence $\tilde{s}_n \in \mathbb{R}$ such that $\tilde{y}_n(\tilde{s}_n) = 1$, $\tilde{y}_n'(\tilde{s}_n) = 0$, and for fixed n there exists a sequence $s_n^j \to \tilde{s}_n$, $j \to \infty$ with $\tilde{y}_n''(s_n^j) \leq 0$).
Setting $s = s_n$ in (5.10) and using (5.15):

$$\epsilon_n^{-\frac{2}{3}} g_{yy}(x_0(\epsilon_n^{\frac{2}{3}} s_n), \tilde{Y}(\epsilon_n^{\frac{2}{3}} s_n))\tilde{y}_n(s_n) \leq \epsilon_n^{-\frac{2}{3}} \tilde{f}_n(s_n), \quad n \geq 1. \tag{5.16}$$

The above relation implies that the sequence $\{s_n\}$ is bounded. Indeed, suppose that for a subsequence $|s_n| \to \infty$. Then $|s_n| \geq D$ for large n. But (5.16) via (5.29), (5.15), yields

$$c \leq \epsilon_n^{-\frac{2}{3}} \tilde{f}_n(s_n) \quad \text{ for large } n$$

which contradicts (5.11).
Passing again to a subsequence, we get $s_n \to \bar{s}$. Since $\tilde{y}_n \to \bar{y}$ in $H^2_{loc}(\mathbb{R})$ and $\tilde{y}_n(s_n) \geq \frac{1}{2}$, we have $\bar{y}(\bar{s}) \geq \frac{1}{2}$. Thus $\bar{y}$ is not identically zero. This combined with (5.13), (5.14) contradicts Proposition 5.1. The "blow up" argument worked and consequently Proposition 5.2 is proved. □

Proposition 5.3. *Let $y \in H^2(\mathbb{R})$, $f \in L^\infty(\mathbb{R})$ with isolated points of discontinuity and*

$$-\epsilon^2 y'' + g_{yy}(x_0(s), \tilde{Y}(s))y = \tilde{Y}(s)f(s), \quad s \in \mathbb{R}.$$

Then if $\epsilon > 0$ is sufficiently small (independent of y, f) we have

$$||y||_{L^\infty(\mathbb{R})} \leq C\epsilon^{-\frac{1}{3}} ||f||_{L^\infty(\mathbb{R})}$$

($C > 0$ independent of y, f, ϵ).

Proof. The proof is similar to that of Proposition 5.2 and makes use of

$$|\tilde{Y}(s)| \leq C(|s|^{\frac{1}{2}} + \epsilon^{\frac{1}{3}}), \quad s \in \mathbb{R}. \tag{5.17}$$

Again arguing by contradiction and re-scaling we have that there exist $\epsilon_n > 0$, $\tilde{y}_n \in H^2(\mathbb{R})$ and $\tilde{f}_n \in L^\infty(\mathbb{R})$ with isolated points of discontinuity, such that

$$-\tilde{y}_n'' + \epsilon_n^{-\frac{2}{3}} g_{yy}(x_0(\epsilon_n^{\frac{2}{3}} s), \tilde{Y}(\epsilon_n^{\frac{2}{3}} s))\tilde{y}_n = \epsilon_n^{-\frac{1}{3}} \tilde{Y}(\epsilon_n^{\frac{2}{3}} s)\epsilon_n^{-\frac{1}{3}} \tilde{f}_n(s), \quad s \in \mathbb{R}, \tag{5.18}$$

$$\epsilon_n \to 0, \quad ||\tilde{y}_n||_{L^\infty(\mathbb{R})} = 1, \quad \epsilon_n^{-\frac{1}{3}} ||\tilde{f}_n||_{L^\infty(\mathbb{R})} \to 0, \quad n \to \infty. \tag{5.19}$$

Relation (5.17) implies that

$$|\epsilon_n^{-\frac{1}{3}} \tilde{Y}(\epsilon_n^{\frac{2}{3}} s)| \leq C(|s|^{\frac{1}{2}} + 1), \quad s \in \mathbb{R}, \ n \geq 1. \tag{5.20}$$

From (5.19), (5.12), (5.20), (5.18) and the usual diagonal argument, we get (passing to a subsequence) that $\tilde{y}_n \overset{H^2_{loc}(\mathbb{R})}{\to} \bar{y}$, $n \to \infty$ with

$$-\bar{y}'' + 2(3aR^2(s) - bms)\bar{y} = 0, \quad s \in \mathbb{R}, \quad ||\bar{y}||_{L^\infty(\mathbb{R})} \leq 1. \tag{5.21}$$

Without loss of generality we may assume that there exist $s_n \in \mathbb{R}$ such that

$$\tilde{y}_n(s_n) \geq \frac{1}{2}, \quad \tilde{y}_n''(s_n) \leq 0. \tag{5.22}$$

We claim that the sequence $\{s_n\}$ is bounded. Indeed, suppose that for a subsequence $|s_n| \to \infty$. Then $|s_n| \geq D$ for large n. Setting $s = s_n$ in (5.18) and using (5.22) and Lemma 5.1:

$$\epsilon_n^{-\frac{2}{3}} g_{yy}(x_0(\epsilon_n^{\frac{2}{3}} s_n), \tilde{Y}(\epsilon_n^{\frac{2}{3}} s_n)) \leq 2\epsilon_n^{-\frac{1}{3}} |\tilde{Y}(\epsilon_n^{\frac{2}{3}} s_n)| \epsilon_n^{-\frac{1}{3}} ||\tilde{f}_n||_{L^\infty(\mathbb{R})}, \quad \text{for large } n. \tag{5.23}$$

If for infinite n:

$$D \leq |s_n| \leq \epsilon_n^{-\frac{2}{3}},$$

then (5.23), (5.2), (5.19), (5.20) yield

$$c|s_n| \leq C(|s_n|^{\frac{1}{2}} + 1), \quad \text{for infinite } n,$$

which contradicts $|s_n| \to \infty$.
If for infinite n:

$$\epsilon_n^{-\frac{2}{3}} \leq |s_n|,$$

then (5.23), (5.3), (5.19) and $||\tilde{Y}||_{L^\infty(\mathbb{R})} \leq C$ imply:

$$c\epsilon_n^{-\frac{2}{3}} \leq \epsilon_n^{-\frac{1}{3}}, \quad \text{for infinite } n,$$

which contradicts $\epsilon_n \to 0$, $n \to \infty$.
Consequently $\{s_n\}$ is bounded which leads to a contradiction as in Proposition 5.2. This proves Proposition 5.3. □

The following Lemma follows from Proposition 5.1 and will be proved in Appendix C.

Lemma 5.2. *There exists a large $K_0 > 0$ such that if $K \geq K_0$, the eigenvalue problem*

$$-y'' + 2(3aR^2(s) - bms)y = \lambda y, \qquad s \in [-K, K],$$

$$y'(-K) = y'(K) = 0$$

has only strictly positive eigenvalues.

Lemma 5.3. *If $\epsilon > 0$ is sufficiently small then $\forall y \in H^1(\mathbb{R})$ the following hold:*

$$\int_{-\infty}^{\infty} \epsilon^2 y'^2 + g_{yy}(x_0(s), \tilde{Y}(s))y^2 ds \geq c\epsilon^{\frac{2}{3}} \int_{-\infty}^{\infty} y^2 ds \tag{5.24}$$

$$\int_{-\infty}^{\infty} \epsilon^2 y'^2 + g_{yy}(x_0(s), \tilde{Y}(s))y^2 ds \geq c \int_{-\infty}^{\infty} \tilde{Y}^2 y^2 ds \tag{5.25}$$

$$\int_{-\infty}^{\infty} \epsilon^2 y'^2 + g_{yy}(x_0(s), \tilde{Y}(s))y^2 ds \geq c \int_{-\infty}^{\infty} y_0^2 y^2 ds \tag{5.26}$$

(c independent of ϵ, y).

Proof. Lemma 5.2 and the variational characterization of the eigenvalues yield

$$\int_{-K_0}^{K_0} \tilde{y}'^2 + 2(3aR^2(s) - bms)\tilde{y}^2 ds \geq c \int_{-K_0}^{K_0} \tilde{y}^2(s) ds \qquad \forall \tilde{y} \in H^1(-K_0, K_0) \tag{5.27}$$

($c > 0$ the principal eigenvalue in $[-K_0, K_0]$)

Let $y \in H^1(-K_0\epsilon^{\frac{2}{3}}, K_0\epsilon^{\frac{2}{3}})$, then $\tilde{y}(s) := y(\epsilon^{\frac{2}{3}} s) \in H^1(-K_0, K_0)$ and

$$\int_{-K_0\epsilon^{\frac{2}{3}}}^{K_0\epsilon^{\frac{2}{3}}} \epsilon^2 {y'}^2 + g_{yy}(x_0(s), \tilde{Y}(s)) y^2 ds = \epsilon^{\frac{2}{3}} \int_{-K_0}^{K_0} \epsilon^{\frac{2}{3}} \tilde{y}'^2 + g_{yy}(x_0(\epsilon^{\frac{2}{3}} s), \tilde{Y}(\epsilon^{\frac{2}{3}} s))\tilde{y}^2(s) ds =$$

$$= \epsilon^{\frac{4}{3}} \int_{-K_0}^{K_0} \tilde{y}'^2 + 2(3aR^2(s) - bms)\tilde{y}^2 ds + \epsilon^{\frac{4}{3}} \int_{-K_0}^{K_0} [\epsilon^{-\frac{2}{3}} g_{yy}(x_0(\epsilon^{\frac{2}{3}} s), \tilde{Y}(\epsilon^{\frac{2}{3}} s)) - 2(3aR^2(s) - bms)]\tilde{y}^2 ds \geq$$

$$\overset{(5.27)}{\geq} c\epsilon^{\frac{4}{3}} \int_{-K_0}^{K_0} \tilde{y}^2 ds - \epsilon^{\frac{4}{3}} \max_{|s| \leq K_0} \left| \epsilon^{-\frac{2}{3}} g_{yy}(x_0(\epsilon^{\frac{2}{3}} s), \tilde{Y}(\epsilon^{\frac{2}{3}} s)) - 2(3aR^2(s) - bms) \right| \int_{-K_0}^{K_0} \tilde{y}^2 ds \geq$$

$$\overset{(5.12)}{\geq} \frac{c}{2} \epsilon^{\frac{4}{3}} \int_{-K_0}^{K_0} \tilde{y}^2 ds = \frac{c}{2} \epsilon^{\frac{2}{3}} \int_{-K_0\epsilon^{\frac{2}{3}}}^{K_0\epsilon^{\frac{2}{3}}} y^2 ds,$$

i.e.

$$\int_{-K_0\epsilon^{\frac{2}{3}}}^{K_0\epsilon^{\frac{2}{3}}} \epsilon^2 {y'}^2 + g_{yy}(x_0(s), \tilde{Y}(s)) y^2 ds \geq c\epsilon^{\frac{2}{3}} \int_{-K_0\epsilon^{\frac{2}{3}}}^{K_0\epsilon^{\frac{2}{3}}} y^2 ds, \quad \forall y \in H^1(-K_0\epsilon^{\frac{2}{3}}, K_0\epsilon^{\frac{2}{3}}). \tag{5.28}$$

Proof of (5.24)
From Lemma 5.1,

$$g_{yy}(x_0(s), \tilde{Y}(s)) \geq c\epsilon^{\frac{2}{3}}, \qquad |s| \geq D\epsilon^{\frac{2}{3}}. \tag{5.29}$$

From here on we assume that $K_0 > D$. Let $y \in H^1(\mathbb{R})$ then

$$\int_{-\infty}^{\infty} \epsilon^2 {y'}^2 + g_{yy}(x_0(s), \tilde{Y}(s)) y^2 ds =$$

$$= \int_{|s| \leq K_0\epsilon^{\frac{2}{3}}} \epsilon^2 {y'}^2 + g_{yy}(x_0(s), \tilde{Y}(s)) y^2 ds + \int_{|s| \geq K_0\epsilon^{\frac{2}{3}}} \epsilon^2 {y'}^2 + g_{yy}(x_0(s), \tilde{Y}(s)) y^2 ds \geq$$

$$\overset{(5.28),(5.29)}{\geq} c\epsilon^{\frac{2}{3}} \int_{|s| \leq K_0\epsilon^{\frac{2}{3}}} y^2 ds + c\epsilon^{\frac{2}{3}} \int_{|s| \geq K_0\epsilon^{\frac{2}{3}}} y^2 ds = c\epsilon^{\frac{2}{3}} \int_{-\infty}^{\infty} y^2 ds.$$

This proves (5.24).
Proof of (5.25)
It follows from Lemma 5.1 and (5.17), that

$$g_{yy}(x_0(s), \tilde{Y}(s)) \geq c\tilde{Y}^2(s), \qquad |s| \geq D\epsilon^{\frac{2}{3}}. \tag{5.30}$$

From (5.17) we obtain

$$|\tilde{Y}(s)| \leq C\epsilon^{\frac{1}{3}}, \qquad |s| \leq K_0\epsilon^{\frac{2}{3}}. \tag{5.31}$$

Let $y \in H^1(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} \epsilon^2 {y'}^2 + g_{yy}(x_0(s), \tilde{Y}(s)) y^2 ds =$$

$$= \int_{|s| \leq K_0\epsilon^{\frac{2}{3}}} \epsilon^2 {y'}^2 + g_{yy}(x_0(s), \tilde{Y}(s)) y^2 ds + \int_{|s| \geq K_0\epsilon^{\frac{2}{3}}} \epsilon^2 {y'}^2 + g_{yy}(x_0(s), \tilde{Y}(s)) y^2 ds \geq$$

$$\overset{(5.28),(5.30)}{\geq} c\epsilon^{\frac{2}{3}}\int_{|s|\leq K_0\epsilon^{\frac{2}{3}}} y^2 ds + c\int_{|s|\geq K_0\epsilon^{\frac{2}{3}}} \tilde{Y}^2 y^2 ds \geq$$

$$\overset{(5.31)}{\geq} c\int_{|s|\leq K_0\epsilon^{\frac{2}{3}}} \tilde{Y}^2 y^2 ds + c\int_{|s|\geq K_0\epsilon^{\frac{2}{3}}} \tilde{Y}^2 y^2 ds.$$

This proves (5.25).

Proof of (5.26)

Relation (5.26) is proved similarly using $y_0(s) = 0$, $s \leq 0$, $g_{yy}(x_0(s), \tilde{Y}(s)) \geq c y_0^2(s)$, $s \geq D\epsilon^{\frac{2}{3}}$ (from Lemma 5.1, (3.14)), and $0 \leq y_0(s) \leq C\epsilon^{\frac{1}{3}}$, $0 \leq s \leq K_0\epsilon^{\frac{2}{3}}$ (from (3.14)). □

Proposition 5.4. *Let $y \in H^2(\mathbb{R})$, $f \in L^2(\mathbb{R})$ and*

$$-\epsilon^2 y'' + g_{yy}(x_0(s), \tilde{Y}(s))y = f \ , \qquad s \in \mathbb{R}.$$

Then if $\epsilon > 0$ is sufficiently small (independent of y, f), the following hold:

$$||y||_{L^2(\mathbb{R})} \leq C\epsilon^{-\frac{2}{3}}||f||_{L^2(\mathbb{R})} \tag{5.32}$$

$$||\tilde{Y}y||_{L^2(\mathbb{R})} \leq C\epsilon^{-\frac{1}{3}}||f||_{L^2(\mathbb{R})} \tag{5.33}$$

$$||y_0 y||_{L^2(\mathbb{R})} \leq C\epsilon^{-\frac{1}{3}}||f||_{L^2(\mathbb{R})} \tag{5.34}$$

with $C > 0$ independent of y, f, ϵ.

Proof. Multiplying the equation by y, integrating by parts ($y \in H^2(\mathbb{R})$) and using the C-S inequality, we obtain

$$\int_{-\infty}^{\infty} \epsilon^2 {y'}^2 + g_{yy}(x_0(s), \tilde{Y}(s))y^2 ds \leq ||y||_{L^2(\mathbb{R})}||f||_{L^2(\mathbb{R})}. \tag{5.35}$$

Relation (5.32) follows from (5.35) via (5.24). Relation (5.33) follows from (5.35) via (5.25), (5.32). Relation (5.34) follows from (5.35) via (5.26), (5.32). The Proposition is proved. □

Proposition 5.5. *Let $y \in H^2(\mathbb{R})$, $f \in L^2(\mathbb{R})$ and*

$$-\epsilon^2 y'' + g_{yy}(x_0(s), \tilde{Y}(s))y = \tilde{Y}f \ , \qquad s \in \mathbb{R}.$$

Then if $\epsilon > 0$ is sufficiently small (independent of y, f), the following hold:

$$||y||_{L^2(\mathbb{R})} \leq C\epsilon^{-\frac{1}{3}}||f||_{L^2(\mathbb{R})} \tag{5.36}$$

$$||\tilde{Y}y||_{L^2(\mathbb{R})} \leq C||f||_{L^2(\mathbb{R})} \tag{5.37}$$

$$||y_0 y||_{L^2(\mathbb{R})} \leq C||f||_{L^2(\mathbb{R})} \tag{5.38}$$

with $C > 0$ independent of y, f, ϵ.

Proof. Multiplying the equation by y, integrating by parts ($y \in H^2(\mathbb{R})$) and using the C-S inequality, we get

$$\int_{-\infty}^{\infty} \epsilon^2 {y'}^2 + g_{yy}(x_0(s), \tilde{Y}(s))y^2 ds \leq ||\tilde{Y}y||_{L^2(\mathbb{R})}||f||_{L^2(\mathbb{R})}, \tag{5.39}$$

and via (5.25):

$$||\tilde{Y}y||_{L^2(\mathbb{R})} \leq C||f||_{L^2(\mathbb{R})},$$

which proves (5.37). Now (5.39) becomes

$$\int_{-\infty}^{\infty} \epsilon^2 y'^2 + g_{yy}(x_0(s), \tilde{Y}(s))y^2 ds \leq C||f||^2_{L^2(\mathbb{R})}. \tag{5.40}$$

Relation (5.36) follows from (5.40) via (5.24). Relation (5.38) follows from (5.40) via (5.26). The Proposition is proved. □

Remark 5.2. *Note that the linear operator* $L_\epsilon y = -\epsilon^2 y'' + g_{yy}(x_0(s), \tilde{Y}(s))y$ *with* $D(L_\epsilon) = H^2(\mathbb{R})$ *is self-adjoint in* $L^2(\mathbb{R})$ *(since* $g_{yy}(x_0(s), \tilde{Y}(s)) \in L^\infty(\mathbb{R})$*). Taking into consideration that* $\lim_{s\to-\infty} g_{yy}(x_0(s), \tilde{Y}(s)) = g_{yy}(0,0) > 0$ *and* $\lim_{s\to\infty} g_{yy}(x_0(s), \tilde{Y}(s)) = g_{yy}(1, \sqrt{2}) > 0$*, we get* $\sigma_{ess}(L_\epsilon) \subset [c, \infty)$ *with* $c = min\{g_{yy}(0,0), g_{yy}(1, \sqrt{2})\} > 0$*. Also (5.32) yields* $Ker(L_\epsilon) = 0$ *for small* $\epsilon > 0$*. From the above, we conclude* $R(L_\epsilon) = L^2(\mathbb{R})$ *provided* $\epsilon > 0$ *is sufficiently small.*

6. Existence of a solution Y_r^ϵ for the "reduced" problem

In this section we will prove the existence of a solution $Y_r^\epsilon \in C^2(\mathbb{R})$ for the "reduced" problem

$$\epsilon^2 y'' = g_y(x_0, y), \qquad s \in \mathbb{R} \tag{6.1}$$

$$y(-\infty) = 0, \qquad y(\infty) = \sqrt{2} \tag{6.2}$$

with $Y_r^\epsilon \overset{\epsilon\to 0}{\to} y_0 \in C(\mathbb{R})$ (recall $g_y(x_0(s), y_0(s)) = 0,\ s \in \mathbb{R}$ and y_0 satisfies (6.2)). We seek $Y_r^\epsilon = Y_{ap}^\epsilon + y$ with $y \in H^2(\mathbb{R})$. Obviously Y_r^ϵ satisfies the desired conditions at $\pm\infty$ (cf. (4.79)). In order to satisfy the equation, we must have for $s \neq \pm d_\epsilon$:

$$-\epsilon^2 y'' + g_{yy}(x_0, Y_{ap})y = -g_y(x_0, Y_{ap}+y) + g_y(x_0, Y_{ap}) + g_{yy}(x_0, Y_{ap})y + \epsilon^2 Y_{ap}'' - g_y(x_0, Y_{ap}),$$

or equivalently

$$-\epsilon^2 y'' + g_{yy}(x_0, Y_{ap})y = -3y^3 - 9Y_{ap}y^2 + \epsilon^2 Y_{ap}'' - g_y(x_0, Y_{ap}), \quad s \neq \pm d_\epsilon. \tag{6.3}$$

Let

$$B_\epsilon = \{y \in H^2(\mathbb{R}) : \ ||y||_{L^\infty(\mathbb{R})} \leq M|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{1}{3}}, \ ||y||_{L^2(\mathbb{R})} \leq M|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{2}{3}}\}$$

with $M > 0$ independent of $\epsilon > 0$ to be determined.

We also define a mapping $T : B_\epsilon \to H^2(\mathbb{R})$ from the relation

$$-\epsilon^2 (Ty)'' + g_{yy}(x_0, Y_{ap})Ty = -3y^3 - 9Y_{ap}y^2 + \epsilon^2 Y_{ap}'' - g_y(x_0, Y_{ap}), \quad s \neq \pm d_\epsilon. \tag{6.4}$$

The fact that T is well defined, follows from the following observations. The differential operator on the left hand side satisfies the hypothesis of Section 5 with $\tilde{Y} = Y_{ap}$ (cf. the definition of $\tilde{Y}, Y_{ap}$ and (4.73), (5.1)). For $y \in H^2(\mathbb{R})$, the right hand side of (6.4) is in $L^2(\mathbb{R})$ (cf. (4.81)) and thus we can use Remark 5.2. Also note that since the right hand side of (6.4) is in $C(\mathbb{R} - \{\pm d_\epsilon\})$ (cf. below (4.79)), we conclude that $Ty \in C^2(\mathbb{R} - \{\pm d_\epsilon\}) \cap C^1(\mathbb{R})$ and

$$-(Ty)''(\pm d_\epsilon^-) + (Ty)''(\pm d_\epsilon^+) = Y_{ap}''(\pm d_\epsilon^-) - Y_{ap}''(\pm d_\epsilon^+)$$

i.e.

$$Ty + Y_{ap} \in C^2(\mathbb{R}). \tag{6.5}$$

Lemma 6.1. *There exists* $M > 0$ *such that if* $\epsilon > 0$ *is sufficiently small, we have* $T : B_\epsilon \to B_\epsilon$.

Proof. Let $y \in B_\epsilon$. Applying Propositions 5.2, 5.3 with $\tilde{Y} = Y_{ap}$ we obtain from (6.4):

$$||Ty||_{L^\infty(\mathbb{R})} \leq C\epsilon^{-\frac{2}{3}}||y^3||_{L^\infty(\mathbb{R})} + C\epsilon^{-\frac{1}{3}}||y^2||_{L^\infty(\mathbb{R})} + C\epsilon^{-\frac{2}{3}}||\epsilon^2 Y_{ap}'' - g_y(x_0, Y_{ap})||_{L^\infty(\mathbb{R})}.$$

Now using $||y||_{L^\infty(\mathbb{R})} \leq M|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{1}{3}}$ and (4.80):

$$\begin{aligned} ||Ty||_{L^\infty(\mathbb{R})} &\leq CM^3|\ln\epsilon|^{-\frac{3}{2}}\epsilon^{\frac{1}{3}} + CM^2|\ln\epsilon|^{-1}\epsilon^{\frac{1}{3}} + C|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{1}{3}} \\ &= C(M^3|\ln\epsilon|^{-1} + M^2|\ln\epsilon|^{-\frac{1}{2}} + 1)|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{1}{3}}. \end{aligned} \tag{6.6}$$

Applying Propositions 5.4, 5.5 with $\tilde{Y} = Y_{ap}$ we get from (6.4):

$$\begin{aligned} ||Ty||_{L^2(\mathbb{R})} &\leq C\epsilon^{-\frac{2}{3}}||y^3||_{L^2(\mathbb{R})} + C\epsilon^{-\frac{1}{3}}||y^2||_{L^2(\mathbb{R})} + C\epsilon^{-\frac{2}{3}}||\epsilon^2 Y_{ap}'' - g_y(x_0, Y_{ap})||_{L^2(\mathbb{R})} \\ &\overset{(4.81)}{\leq} C\epsilon^{-\frac{2}{3}}||y^2||_{L^\infty(\mathbb{R})}||y||_{L^2(\mathbb{R})} + C\epsilon^{-\frac{1}{3}}||y||_{L^\infty(\mathbb{R})}||y||_{L^2(\mathbb{R})} + C|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{2}{3}} \\ &\overset{y\in B_\epsilon}{\leq} CM^3|\ln\epsilon|^{-\frac{3}{2}}\epsilon^{\frac{2}{3}} + CM^2|\ln\epsilon|^{-1}\epsilon^{\frac{2}{3}} + C|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{2}{3}}, \end{aligned}$$

i.e.

$$||Ty||_{L^2(\mathbb{R})} \leq C(M^3|\ln\epsilon|^{-1} + M^2|\ln\epsilon|^{-\frac{1}{2}} + 1)|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{2}{3}}. \tag{6.7}$$

Relations (6.6), (6.7) imply that if we choose $M = 2C$, then $||Ty||_{L^\infty(\mathbb{R})} \leq M|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{1}{3}}$ and $||Ty||_{L^2(\mathbb{R})} \leq M|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{2}{3}}$ provided $\epsilon > 0$ is sufficiently small. Thus $Ty \in B_\epsilon$ which proves the Lemma. □

Lemma 6.2. *If $\epsilon > 0$ is sufficiently small, there exists $y_* \in B_\epsilon$ such that $Ty_* = y_*$.*

Proof. First we show that $T : B_\epsilon \to B_\epsilon$ is a contraction in the $L^2(\mathbb{R})$ norm. Indeed, let $y_1,\ y_2 \in B_\epsilon$ then (6.4) yields

$$-\epsilon^2(Ty_1 - Ty_2)'' + g_{yy}(x_0, Y_{ap})(Ty_1 - Ty_2) = -3(y_1^3 - y_2^3) - 9Y_{ap}(y_1^2 - y_2^2),$$

i.e.

$$L_\epsilon(Ty_1 - Ty_2) = -3(y_1^2 + y_1y_2 + y_2^2)(y_1 - y_2) - 9Y_{ap}(y_1 + y_2)(y_1 - y_2).$$

Applying Propositions 5.4, 5.5 with $\tilde{Y} = Y_{ap}$ and since $||y_i||_{L^\infty(\mathbb{R})} \leq M|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{1}{3}},\ i = 1, 2$, we get

$$||Ty_1 - Ty_2||_{L^2(\mathbb{R})} \leq C(|\ln\epsilon|^{-1} + |\ln\epsilon|^{-\frac{1}{2}})||y_1 - y_2||_{L^2(\mathbb{R})}.$$

Thus if $\epsilon > 0$ is sufficiently small:

$$||Ty_1 - Ty_2||_{L^2(\mathbb{R})} \leq \frac{1}{2}||y_1 - y_2||_{L^2(\mathbb{R})}, \quad \forall y_1, y_2 \in B_\epsilon. \tag{6.8}$$

We define a sequence $y_n \in B_\epsilon,\ n \geq 0$, by $y_0 = 0,\ y_{n+1} = Ty_n,\ n \geq 0$. From (6.8) we get that $\{y_n\}$ is Cauchy in $L^2(\mathbb{R})$. For $n \geq 0$,

$$-\epsilon^2 y_{n+1}'' + g_{yy}(x_0, Y_{ap})y_{n+1} = -3y_n^3 - 9Y_{ap}y_n^2 + \epsilon^2 Y_{ap}'' - g_y(x_0, Y_{ap}) \tag{6.9}$$

and using $||y_n||_{L^\infty(\mathbb{R})} \leq C,\ n \geq 0$ (this suffices) we obtain,

$$||y_{n+m}'' - y_n''||_{L^2(\mathbb{R})} \leq \frac{C}{\epsilon^2}\left(||y_{n+m} - y_n||_{L^2(\mathbb{R})} + ||y_{n+m-1} - y_{n-1}||_{L^2(\mathbb{R})}\right), \quad n, m \geq 1.$$

Since $||y_{n+m} - y_n||_{L^2(\mathbb{R})} \overset{n\to\infty}{\to} 0$, the above relation and an interpolation inequality yield $||y_{n+m} - y_n||_{H^2(\mathbb{R})} \overset{n\to\infty}{\to} 0$. Consequently, there exists $y_* \in B_\epsilon$ such that $y_n \overset{H^2(\mathbb{R})}{\to} y_*$ as $n \to \infty$. Taking limits $n \to \infty$ in (6.9), yields $Ty_* = y_*$. This proves the Lemma. □

Theorem 6.1. *There exists a solution $Y_r^\epsilon \in C^2(\mathbb{R})$ of the "reduced" problem (6.1), (6.2) such that*

$$||Y_r - y_0||_{L^\infty(\mathbb{R})} \le C\epsilon^{\frac{1}{3}}, \tag{6.10}$$

$$||Y_r - y_0||_{L^2(\mathbb{R})} \le C\epsilon^{\frac{2}{3}}, \tag{6.11}$$

$$||y_0(Y_r - y_0)||_{L^2(\mathbb{R})} \le C\epsilon, \tag{6.12}$$

$$||Y_r(Y_r - y_0)||_{L^2(\mathbb{R})} \le C\epsilon. \tag{6.13}$$

Proof. From Lemmas 6.1, 6.2 and (6.5) we get that $Y_r^\epsilon := Y_{ap}^\epsilon + y_*^\epsilon \in C^2(\mathbb{R})$ solves the "reduced" problem (6.1), (6.2).
Relations (6.10), (6.11) follow from the fact that $y_* \in B_\epsilon$ and (4.82), (4.83).
Since $Ty_* = y_*$ and $y_* \in B_\epsilon$, we obtain from (6.4) via (5.34), (5.38), (4.81) that

$$||y_0 y_*||_{L^2(\mathbb{R})} \le C|\ln\epsilon|^{-\frac{1}{2}}\epsilon.$$

Relation (6.12) follows from the above relation and (4.84). Relation (6.13) follows from (6.10), (6.11) and (6.12). The Theorem is proved. □

Remark 6.1. *Since $Y_r := y_{ap} + (\hat\rho + y_*)$ with $||\hat\rho + y_*||_{L^\infty(\mathbb{R})} \le C|\ln\epsilon|^{-\frac{1}{2}}\epsilon^{\frac{1}{3}}$, we can apply the results of Section 5 with $\tilde{Y} = Y_r$.*

7. The linear operator K_ϵ

We define a linear operator $K : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ from the relation (cf. Remarks 5.2, 6.1):

$$-\epsilon^2(Kx)'' + g_{yy}(x_0, Y_r)Kx = -g_{xy}(x_0, Y_r)x, \qquad s \in \mathbb{R}. \tag{7.1}$$

Since $g_{xy}(x_0, Y_r) = 4Y_r(x_0(s) - 3)$, Propositions 5.3, 5.5 with $\tilde{Y} = Y_r$ give

$$||Kx||_{L^\infty(\mathbb{R})} \le C\epsilon^{-\frac{1}{3}}||x||_{L^\infty(\mathbb{R})}, \tag{7.2}$$

$$||Kx||_{L^2(\mathbb{R})} \le C\epsilon^{-\frac{1}{3}}||x||_{L^2(\mathbb{R})}. \tag{7.3}$$

Multiplying (7.1) by s, integrating over $[0, D\epsilon^{\frac{2}{3}}]$ and using (7.2), we obtain with the same procedure as we did for (4.34):

$$|(Kx)'(D\epsilon^{\frac{2}{3}})| \le C\epsilon^{-1}||x||_{L^\infty(\mathbb{R})}. \tag{7.4}$$

Let (cf. (4.31))

$$w(s) = Kx + \frac{g_{xy}(x_0, y_0)}{g_{yy}(x_0, y_0)}x, \qquad s \ne 0. \tag{7.5}$$

The following estimates are easy consequences of (3.10), (3.13), (3.14), (4.31).

$$\left|\frac{g_{xy}(x_0, y_0)}{g_{yy}(x_0, y_0)}\right| \le C\epsilon^{-\frac{1}{3}}, \qquad s \ge D\epsilon^{\frac{2}{3}}, \tag{7.6}$$

$$\left|\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}\right)'\right| \leq Cs^{-\frac{3}{2}}, \quad \left|\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}\right)''\right| \leq Cs^{-\frac{5}{2}}, \quad s>0. \tag{7.7}$$

Relations (7.2), (7.3) via (7.6) yield

$$||w||_{L^\infty(D\epsilon^{\frac{2}{3}},\infty)} \leq C\epsilon^{-\frac{1}{3}}||x||_{L^\infty(\mathbb{R})}, \tag{7.8}$$

$$||w||_{L^2(D\epsilon^{\frac{2}{3}},\infty)} \leq C\epsilon^{-\frac{1}{3}}||x||_{L^2(\mathbb{R})}. \tag{7.9}$$

Also from (7.4), (7.6), (7.7) we have

$$|w'(D\epsilon^{\frac{2}{3}})| \leq C\epsilon^{-1}||x||_{H^2(\mathbb{R})}. \tag{7.10}$$

Lemma 7.1. *If $\epsilon>0$ is sufficiently small, we have*

$$||g_{xy}(x_0,y_0)w||_{L^2(D\epsilon^{\frac{2}{3}},\infty)} \leq C\epsilon^{\frac{1}{6}}||x||_{H^2(\mathbb{R})}, \quad \forall x \in H^2(\mathbb{R}),$$

($C>0$ independent of x,ϵ).

Proof. We observe that Lemma 5.1 with $\tilde{Y}=Y_r$ implies $g^2_{xy}(x_0,y_0) \leq Cg_{yy}(x_0,Y_r), \ s \geq D\epsilon^{\frac{2}{3}}$. Thus it suffices to show

$$||g^{\frac{1}{2}}_{yy}(x_0,Y_r)w||_{L^2(D\epsilon^{\frac{2}{3}},\infty)} \leq C\epsilon^{\frac{1}{6}}||x||_{H^2(\mathbb{R})}, \quad \forall x \in H^2(\mathbb{R}). \tag{7.11}$$

Using (7.1) we see that w satisfies for $s \geq D\epsilon^{\frac{2}{3}}$,

$$-\epsilon^2 w''+g_{yy}(x_0,Y_r)w = -\epsilon^2\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x\right)''+\left(g_{yy}(x_0,Y_r)\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}-g_{xy}(x_0,Y_r)\right)x.$$

Multiplying the above equation by w and integrating by parts over $[D\epsilon^{\frac{2}{3}},\infty)$:

$$\epsilon^2\int_{D\epsilon^{\frac{2}{3}}}^\infty w'^2 ds+\epsilon^2 w'(D\epsilon^{\frac{2}{3}})w(D\epsilon^{\frac{2}{3}})+\int_{D\epsilon^{\frac{2}{3}}}^\infty g_{yy}(x_0,Y_r)w^2 ds =$$

$$= -\epsilon^2\int_{D\epsilon^{\frac{2}{3}}}^\infty \left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x\right)'' w ds+\int_{D\epsilon^{\frac{2}{3}}}^\infty \left(g_{yy}(x_0,Y_r)\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}-g_{xy}(x_0,Y_r)\right)xw ds,$$

or via (7.8), (7.10)

$$\begin{aligned}\int_{D\epsilon^{\frac{2}{3}}}^\infty g_{yy}(x_0,Y_r)w^2 ds \quad &\leq \quad C\epsilon^{\frac{2}{3}}||x||^2_{H^2(\mathbb{R})}+\epsilon^2\int_{D\epsilon^{\frac{2}{3}}}^\infty \left|\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x\right)''\right||w|ds+ \\ &+\int_{D\epsilon^{\frac{2}{3}}}^\infty \left|g_{yy}(x_0,Y_r)\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}-g_{xy}(x_0,Y_r)\right||x||w|ds.\end{aligned} \tag{7.12}$$

We have

$$\int_{D\epsilon^{\frac{2}{3}}}^\infty \left|\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x\right)''\right||w|ds \leq$$

$$\leq \int_{D\epsilon^{\frac{2}{3}}}^\infty \left|\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}\right||x''||w|ds+2\int_{D\epsilon^{\frac{2}{3}}}^\infty \left|\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}\right)'\right||x'||w|ds+\int_{D\epsilon^{\frac{2}{3}}}^\infty \left|\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}\right)''\right||x||w|ds \leq$$

$$\overset{(7.6),(7.7)}{\leq} C\epsilon^{-\frac{1}{3}}\int_{D\epsilon^{\frac{2}{3}}}^\infty |x''||w|ds+C\int_{D\epsilon^{\frac{2}{3}}}^\infty s^{-\frac{3}{2}}|x'||w|ds+C\int_{D\epsilon^{\frac{2}{3}}}^\infty s^{-\frac{5}{2}}|x||w|ds \leq$$

$$\leq C\epsilon^{-\frac{1}{3}}||x''||_{L^2(D\epsilon^{\frac{2}{3}},\infty)}||w||_{L^2(D\epsilon^{\frac{2}{3}},\infty)}+C||x||_{H^2(\mathbb{R})}||w||_{L^\infty(D\epsilon^{\frac{2}{3}},\infty)}\int_{D\epsilon^{\frac{2}{3}}}^\infty (s^{-\frac{5}{2}}+s^{-\frac{3}{2}})ds \leq$$

$$\overset{(7.8),(7.9)}{\leq} C\epsilon^{-\frac{2}{3}}||x||^2_{H^2(\mathbb{R})} + C\epsilon^{-\frac{4}{3}}||x||^2_{H^2(\mathbb{R})} \leq C\epsilon^{-\frac{4}{3}}||x||^2_{H^2(\mathbb{R})}. \tag{7.13}$$

Also for $s \geq D\epsilon^{\frac{2}{3}}$:

$$\begin{aligned}\left|g_{yy}(x_0,Y_r)\tfrac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)} - g_{xy}(x_0,Y_r)\right| \leq & \ |g_{yy}(x_0,Y_r) - g_{yy}(x_0,y_0)| \left|\tfrac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}\right| + |g_{xy}(x_0,y_0) - g_{xy}(x_0,Y_r)| \\ \overset{(2.6),(7.6)}{\leq} & \ C|Y_r^2 - y_0^2|\epsilon^{-\frac{1}{3}} + C|Y_r - y_0| \\ \leq & \ C|Y_r(Y_r - y_0)|\epsilon^{-\frac{1}{3}} + C|y_0(Y_r - y_0)|\epsilon^{-\frac{1}{3}} + C|Y_r - y_0|.\end{aligned}$$

Now using (6.11), (6.12), (6.13), we obtain that

$$\left|\left|g_{yy}(x_0,Y_r)\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)} - g_{xy}(x_0,Y_r)\right|\right|_{L^2(D\epsilon^{\frac{2}{3}},\infty)} \leq C\epsilon^{\frac{2}{3}}.$$

Thus the C-S inequality and (7.9) yield

$$\int_{D\epsilon^{\frac{2}{3}}}^{\infty}\left|g_{yy}(x_0,Y_r)\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)} - g_{xy}(x_0,Y_r)\right||x||w|ds \leq C\epsilon^{\frac{1}{3}}||x||^2_{H^2(\mathbb{R})}. \tag{7.14}$$

Relation (7.11) follows from (7.12), (7.13), (7.14). The Lemma is proved. □

Lemma 7.2. *If $\epsilon > 0$ is sufficiently small, then*

$$||g_{xy}(x_0,y_0)w||_{L^2(0,D\epsilon^{\frac{2}{3}})} \leq C\epsilon^{\frac{1}{3}}||x||_{L^\infty(\mathbb{R})}, \quad \forall x \in H^2(\mathbb{R}),$$

($C > 0$ independent of x, ϵ).

Proof. For $0 < s \leq D\epsilon^{\frac{2}{3}}$:

$$\begin{aligned}|g_{xy}(x_0,y_0)w| \quad = & \quad \left|g_{xy}(x_0,y_0)Kx + \frac{g^2_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x\right| \\ \overset{(3.14),(4.47)}{\leq} & \quad Cs^{\frac{1}{2}}||Kx||_{L^\infty(\mathbb{R})} + C||x||_{L^\infty(\mathbb{R})} \\ \overset{(7.2)}{\leq} & \quad C\epsilon^{\frac{1}{3}}\epsilon^{-\frac{1}{3}}||x||_{L^\infty(\mathbb{R})} + C||x||_{L^\infty(\mathbb{R})} \leq C||x||_{L^\infty(\mathbb{R})}.\end{aligned}$$

The claim of the Lemma follows. □

Proposition 7.1. *If $\epsilon > 0$ is sufficiently small, then*

$$\left|\left|g_{xy}(x_0,y_0)\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x + Kx\right)\right|\right|_{L^2(\mathbb{R})} \leq C\epsilon^{\frac{1}{6}}||x||_{H^2(\mathbb{R})}, \quad \forall x \in H^2(\mathbb{R}),$$

($C > 0$ independent of x, ϵ).

Proof. We will show (cf. (7.5)) that

$$||g_{xy}(x_0,y_0)w||_{L^2(\mathbb{R})} \leq C\epsilon^{\frac{1}{6}}||x||_{H^2(\mathbb{R})}, \quad \forall x \in H^2(\mathbb{R}).$$

But this follows from Lemmas 7.1, 7.2 and the fact that $g_{xy}(x_0,y_0) = 0, \ s \leq 0$. The Proposition is proved. □

8. The linear operator B

We define a linear operator B with $D(B) = H^2(\mathbb{R})$ and $Bx := -x'' + G_{xx}(x_0(s))x$, G as in (3.5). Observe that $G_{xx}(x_0(s)) \in C^\infty(\mathbb{R} - \{0\})$ and has a finite discontinuity at $s = 0$,

$$G_{xx}(x_0(s)) = g_{xx}(x_0, y_0) - \frac{g_{xy}^2(x_0, y_0)}{g_{yy}(x_0, y_0)}, \quad s \neq 0.$$

We have that B is selfadjoint in $L^2(\mathbb{R})$ and $\sigma_{ess}(B) \subset [c, \infty)$ with $c = \min\{G_{xx}(0), G_{xx}(1)\} > 0$ (cf. (3.6)). Recall that

$$x_0''(s) = G_x(x_0(s)), \quad s \in \mathbb{R} \tag{8.1}$$

and thus $-x_0'''(s) + G_{xx}(x_0(s))x_0'(s) = 0, \quad s \neq 0$, i.e $x_0' \in Ker(B)$ (recall that $x_0' \in H^2(\mathbb{R})$, cf. below (3.10)).

We claim that $dim(Ker(B)) = 1$. Indeed, suppose $\chi \in Ker(B)$. Since $Bx_0' = 0, B\chi = 0$, we obtain $x_0''\chi - x_0'\chi' = c \in \mathbb{R}, \ \ s \in \mathbb{R}$. But $x_0', x_0'', \chi, \chi' \to 0$, as $|s| \to \infty$ and thus $c = 0$, i.e. x_0', χ are linearly dependent.

We also claim that 0 is the principal eigenvalue of B. Indeed, it suffices to show (via a density argument) that

$$\int_{-\infty}^{\infty} {x'}^2 + G_{xx}(x_0(s))x^2 ds \geq 0, \qquad \forall x \in C_c^\infty(\mathbb{R}).$$

Let $x \in C_c^\infty(\mathbb{R})$, then $x = \varphi x_0'$ with $\varphi \in C_c^1(\mathbb{R})$ (since $x_0'(s) > 0, \ s \in \mathbb{R}$) and thus

$$\int_{-\infty}^{\infty} {x'}^2 + G_{xx}(x_0(s))x^2 ds \overset{(8.1)}{=} \int_{-\infty}^{\infty} [\varphi' x_0' + \varphi G_x(x_0(s))]^2 + G_{xx}(x_0(s))\varphi^2 x_0'^2 ds =$$

$$\overset{(8.1)}{=} \int_{-\infty}^{\infty} {\varphi'}^2 {x_0'}^2 + \varphi^2 G_x(x_0(s))x_0'' + 2\varphi'\varphi G_x(x_0(s))x_0' + \varphi^2 G_{xx}(x_0(s))x_0' x_0' ds =$$

$$= \int_{-\infty}^{\infty} {\varphi'}^2 {x_0'}^2 ds + \int_{-\infty}^{\infty} \left(\varphi^2 G_x(x_0(s))x_0'\right)' ds \overset{\varphi \in C_c^1(\mathbb{R})}{\geq} 0$$

Taking the above claims into consideration, we get

Proposition 8.1. *Let $f \in L^2(\mathbb{R})$ with $\int_{-\infty}^{\infty} f x_0' ds = 0$, then there exists a unique $x \in H^2(\mathbb{R})$ with $\int_{-\infty}^{\infty} x x_0' ds = 0$ such that $Bx = f$, i.e.*

$$-x'' + \left(g_{xx}(x_0, y_0) - \frac{g_{xy}^2(x_0, y_0)}{g_{yy}(x_0, y_0)} \right) x = f, \qquad s \in \mathbb{R}.$$

Furthermore,

$$||x||_{H^2(\mathbb{R})} \leq C||f||_{L^2(\mathbb{R})},$$

($C > 0$ independent of f).

9. Existence of a heteroclinic orbit for the system

We seek $x, y \in H^2(\mathbb{R})$ such that $x_0 + x, Y_r + y$ satisfy (2.1) (they obviously satisfy (2.2), (2.3)). We must have $\forall s \in \mathbb{R}$:

$$\begin{aligned} (x_0 + x)'' &= g_x(x_0 + x, Y_r + y) \\ \epsilon^2 (Y_r + y)'' &= g_y(x_0 + x, Y_r + y) \end{aligned}$$

Rearranging terms and using $x_0'' = g_x(x_0, y_0)$, $\epsilon^2 Y_r'' = g_y(x_0, Y_r)$,

$$x'' - \left(g_{xx}(x_0,y_0) - \frac{g_{xy}^2(x_0,y_0)}{g_{yy}(x_0,y_0)}\right)x = \begin{array}{l} g_x(x_0+x, y_0+y) - g_x(x_0,y_0) - g_{xx}(x_0,y_0)x - g_{xy}(x_0,y_0)y + \\ + g_{xy}(x_0,y_0)\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x + y\right) + \\ + g_x(x_0+x, Y_r+y) - g_x(x_0+x, y_0+y) \end{array}$$

$$\epsilon^2 y'' - g_{yy}(x_0,Y_r)y = g_y(x_0+x, Y_r+y) - g_y(x_0,Y_r) - g_{xy}(x_0,Y_r)x - g_{yy}(x_0,Y_r)y + g_{xy}(x_0,Y_r)x.$$

Equivalently

$$-Bx = N_1(x,y) + g_{xy}(x_0,y_0)\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x + y\right) + E_1(x,y)$$

$$-L_\epsilon y = N_2(x,y) + g_{xy}(x_0,Y_r)x$$

with

$$N_1(x,y) = -6y^2 + 2xy^2 + 4y_0xy + 2x_0y^2 + 4x^3 + 12x_0x^2, \tag{9.1}$$

$$E_1(x,y) = (2(x+x_0) - 6)\left((Y_r - y_0)^2 + 2(y+y_0)(Y_r - y_0)\right), \tag{9.2}$$

$$N_2(x,y) = -12xy + 2x^2y + 4x_0xy + 2Y_rx^2 + 3y^3 + 9Y_ry^2. \tag{9.3}$$

Let (as always $\perp$ means in $L^2(\mathbb{R})$)

$$\mathbf{B}_\epsilon = \{(x,y) \in H^2(\mathbb{R}) \times H^2(\mathbb{R}),\ x \perp x_0',\ ||x||_{H^2(\mathbb{R})} \le M_1\epsilon, ||y||_{L^\infty(\mathbb{R})} \le \epsilon^{\frac{1}{2}}, ||y||_{L^2(\mathbb{R})} \le M_2\epsilon^{\frac{2}{3}}\}$$

with $M_1,\ M_2$ independent of $\epsilon > 0$ to be determined.

We define a mapping $\mathbf{T} : \mathbf{B}_\epsilon \to \{(x,y) \in H^2(\mathbb{R}) \times H^2(\mathbb{R}),\ x \perp x_0'\}$, with $\mathbf{T}(x,y) = (\bar{x}, \bar{y})$ and

$$-B\bar{x} = -b(x,y)x_0' + N_1(x,y) + g_{xy}(x_0,y_0)\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x + \bar{y}\right) + E_1(x,y) \tag{9.4}$$

$$-L_\epsilon \bar{y} = N_2(x,y) + g_{xy}(x_0,Y_r)x \tag{9.5}$$

$$b(x,y) = \frac{1}{||x_0'||^2_{L^2(\mathbb{R})}}\left(N_1(x,y) + g_{xy}(x_0,y_0)\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x + \bar{y}\right) + E_1(x,y), x_0'\right)_{L^2(\mathbb{R})}. \tag{9.6}$$

The mapping $\mathbf{T}$ is well defined. Indeed,

Step 1

For $(x,y) \in \mathbf{B}_\epsilon$ the right hand side of (9.5) is in $L^2(\mathbb{R})$ and thus via Remark 5.2 we obtain a unique $\bar{y} \in H^2(\mathbb{R})$.

Step 2

Relation (9.6) implies that the right hand side of (9.4) is normal to x_0'. Consequently via Proposition 8.1 we get a unique $\bar{x} \in H^2(\mathbb{R})$ with $\bar{x} \perp x_0'$.

Lemma 9.1. *There exist large $M_1, M_2 > 0$ such that if $\epsilon > 0$ is sufficiently small, we have $\boldsymbol{T} : \boldsymbol{B}_\epsilon \to \boldsymbol{B}_\epsilon$.*

Proof. Let $(x,y) \in \mathbf{B}_\epsilon$ and $(\bar{x}, \bar{y}) = \mathbf{T}(x,y)$.

Claim 1

$$||\bar{y}||_{L^\infty(\mathbb{R})} \le o_\epsilon(1)\epsilon^{\frac{1}{2}}, \quad ||\bar{y}||_{L^2(\mathbb{R})} \le C(M_1 + o_\epsilon(1))\epsilon^{\frac{2}{3}},$$

where $o_\epsilon(1)\to 0$ as $\epsilon\to 0$ and $C>0$ independent of ϵ, M_1, M_2. ($o_\epsilon(1)$ depends on M_1, M_2).
From (9.3), (9.5) we have

$$-L_\epsilon \bar{y} = (-12+2x+4x_0)xy+2Y_r x^2+3y^3+9Y_r y^2+4Y_r(x_0-3)x. \tag{9.7}$$

From Propositions 5.2, 5.3 (with $\tilde{Y}=Y_r$):

$$\begin{aligned}||\bar{y}||_{L^\infty(\mathbb{R})}\leq\ & C\epsilon^{-\frac{2}{3}}||x||_{L^\infty(\mathbb{R})}||y||_{L^\infty(\mathbb{R})}+C\epsilon^{-\frac{1}{3}}||x||^2_{L^\infty(\mathbb{R})}+C\epsilon^{-\frac{2}{3}}||y||^3_{L^\infty(\mathbb{R})}+\\ &+C\epsilon^{-\frac{1}{3}}||y||^2_{L^\infty(\mathbb{R})}+C\epsilon^{-\frac{1}{3}}||x||_{L^\infty(\mathbb{R})}\end{aligned}$$

and since $(x,y)\in\mathbf{B}_\epsilon$,

$$||\bar{y}||_{L^\infty(\mathbb{R})}\leq CM_1\epsilon^{\frac{5}{6}}+CM_1^2\epsilon^{\frac{5}{3}}+C\epsilon^{\frac{5}{6}}+C\epsilon^{\frac{2}{3}}+CM_1\epsilon^{\frac{2}{3}}. \tag{9.8}$$

Also (9.7) via Propositions 5.4, 5.5 yields

$$||\bar{y}||_{L^2(\mathbb{R})}\leq C\epsilon^{-\frac{2}{3}}||x||_{L^\infty(\mathbb{R})}||y||_{L^2(\mathbb{R})}+C\epsilon^{-\frac{1}{3}}||x||_{L^\infty(\mathbb{R})}||x||_{L^2(\mathbb{R})}+C\epsilon^{-\frac{2}{3}}||y||^2_{L^\infty(\mathbb{R})}||y||_{L^2(\mathbb{R})}+$$

$$+C\epsilon^{-\frac{1}{3}}||y||_{L^\infty(\mathbb{R})}||y||_{L^2(\mathbb{R})}+C\epsilon^{-\frac{1}{3}}||x||_{L^2(\mathbb{R})},$$

and since $(x,y)\in\mathbf{B}_\epsilon$,

$$||\bar{y}||_{L^2(\mathbb{R})}\leq CM_1M_2\epsilon+CM_1^2\epsilon^{\frac{5}{3}}+CM_2\epsilon+CM_2\epsilon^{\frac{5}{6}}+CM_1\epsilon^{\frac{2}{3}}. \tag{9.9}$$

Claim 1 follows from (9.8), (9.9).
<u>Claim 2</u>

$$\left|\left|g_{xy}(x_0,y_0)\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x+\bar{y}\right)\right|\right|_{L^2(\mathbb{R})}\leq o_\epsilon(1)\epsilon.$$

Indeed, from (9.5), (7.1) we get $\bar{y}=-L_\epsilon^{-1}N_2(x,y)+Kx$. By applying Propositions 5.4, 5.5 and noting that $|g_{xy}(x_0,y_0)|\leq C|y_0|$, $s\in\mathbb{R}$, we have as for (9.9) (without including the last term):

$$||g_{xy}(x_0,y_0)L_\epsilon^{-1}N_2(x,y)||_{L^2(\mathbb{R})}\leq C(M_2+o_\epsilon(1))\epsilon^{\frac{7}{6}}.$$

Also Proposition 7.1 yields

$$\left|\left|g_{xy}(x_0,y_0)\left(\frac{g_{xy}(x_0,y_0)}{g_{yy}(x_0,y_0)}x+Kx\right)\right|\right|_{L^2(\mathbb{R})}\leq C\epsilon^{\frac{1}{6}}||x||_{H^2(\mathbb{R})}\leq CM_1\epsilon^{\frac{7}{6}}.$$

Claim 2 follows from the above two relations.
<u>Claim 3</u>

$$||\bar{x}||_{H^2(\mathbb{R})}\leq C(1+o_\epsilon(1))\epsilon.$$

Using that $(x,y)\in\mathbf{B}_\epsilon$ and the estimates of Theorem 6.1, we obtain

$$||N_1(x,y)||_{L^2(\mathbb{R})}\leq o_\epsilon(1)\epsilon \quad\text{and}\quad ||E_1(x,y)||_{L^2(\mathbb{R})}\leq C(1+o_\epsilon(1))\epsilon.$$

Thus applying Proposition 8.1 in (9.4) and taking into consideration the above relation, Claim 2 and (9.6), we easily obtain Claim 3.
Claims 1,3 imply that by choosing $M_1=2C,\ M_2=C(2C+1)$ and provided $\epsilon>0$ is sufficiently small, we have $(\bar{x},\bar{y})\in\mathbf{B}_\epsilon$. The Lemma is proved. □

Lemma 9.2. *If $\epsilon>0$ is sufficiently small, there exists $(x_*,y_*)\in\boldsymbol{B}_\epsilon$ such that $\boldsymbol{T}(x_*,y_*)=(x_*,y_*)$.*

Proof. Let $(x_1, y_1),\ (x_2, y_2) \in \mathbf{B}_\epsilon$. From (9.4), (9.5), (9.6), with $(\bar{x}_i, \bar{y}_i) = \mathbf{T}(x_i, y_i)$ and after simple calculations as in Lemma 9.1:

$$(9.10)\qquad \begin{aligned} ||\bar{y}_1 - \bar{y}_2||_{L^\infty(\mathbb{R})} &\leq C\epsilon^{\frac{1}{6}}||y_1 - y_2||_{L^\infty(\mathbb{R})} + C\epsilon^{-\frac{2}{6}}||x_1 - x_2||_{H^2(\mathbb{R})} \\ ||\bar{x}_1 - \bar{x}_2||_{H^2(\mathbb{R})} &\leq C\epsilon^{\frac{4}{6}}||y_1 - y_2||_{L^\infty(\mathbb{R})} + C\epsilon^{\frac{1}{6}}||x_1 - x_2||_{H^2(\mathbb{R})} \end{aligned}$$

and

$$(9.11)\qquad ||\bar{y}_1 - \bar{y}_2||_{L^2(\mathbb{R})} \leq C\epsilon^{\frac{2}{6}}||y_1 - y_2||_{L^\infty(\mathbb{R})} + C\epsilon^{-\frac{2}{6}}||x_1 - x_2||_{H^2(\mathbb{R})}$$

$$(9.12)\qquad ||\bar{y}_1 - \bar{y}_2||_{H^2(\mathbb{R})} \leq C(\epsilon)||y_1 - y_2||_{L^\infty(\mathbb{R})} + C(\epsilon)||x_1 - x_2||_{H^2(\mathbb{R})}.$$

We will use the notation $(\bar{x}^{(j+1)}, \bar{y}^{(j+1)}) = \mathbf{T}(\bar{x}^j, \bar{y}^j),\ j \geq 1,\ (\bar{x}^1, \bar{y}^1) = (\bar{x}, \bar{y}) = \mathbf{T}(x, y),\ \forall (x, y) \in \mathbf{B}_\epsilon$, i.e. $(\bar{x}^j, \bar{y}^j) = \mathbf{T}^j(x, y),\ \forall (x, y) \in \mathbf{B}_\epsilon$. From (9.10), we have that for $j \geq 1$:

$$\begin{aligned} ||\bar{y}_1^j - \bar{y}_2^j||_{L^\infty(\mathbb{R})} &\leq C(j)\epsilon^{\frac{j}{6}}||y_1 - y_2||_{L^\infty(\mathbb{R})} + C(j)\epsilon^{\frac{j-3}{6}}||x_1 - x_2||_{H^2(\mathbb{R})} \\ ||\bar{x}_1^j - \bar{x}_2^j||_{H^2(\mathbb{R})} &\leq C(j)\epsilon^{\frac{3+j}{6}}||y_1 - y_2||_{L^\infty(\mathbb{R})} + C(j)\epsilon^{\frac{j}{6}}||x_1 - x_2||_{H^2(\mathbb{R})} \end{aligned}$$

and thus via (9.11)

$$||\bar{y}_1^j - \bar{y}_2^j||_{L^2(\mathbb{R})} \leq C(j)\epsilon^{\frac{j}{6}}||y_1 - y_2||_{L^\infty(\mathbb{R})} + C(j)\epsilon^{\frac{j-3}{6}}||x_1 - x_2||_{H^2(\mathbb{R})}$$

Combining the above two relations with (9.11), (9.12), gives

$$(9.13)\qquad \begin{aligned} ||\bar{y}_1^4 - \bar{y}_2^4||_{L^\infty(\mathbb{R})} &\leq C\epsilon^{\frac{4}{6}}||y_1 - y_2||_{L^\infty(\mathbb{R})} + C\epsilon^{\frac{1}{6}}||x_1 - x_2||_{H^2(\mathbb{R})}, \\ ||\bar{x}_1^4 - \bar{x}_2^4||_{H^2(\mathbb{R})} &\leq C\epsilon^{\frac{7}{6}}||y_1 - y_2||_{L^\infty(\mathbb{R})} + C\epsilon^{\frac{4}{6}}||x_1 - x_2||_{H^2(\mathbb{R})}, \\ ||\bar{y}_1^4 - \bar{y}_2^4||_{L^2(\mathbb{R})} &\leq C\epsilon^{\frac{4}{6}}||y_1 - y_2||_{L^\infty(\mathbb{R})} + C\epsilon^{\frac{1}{6}}||x_1 - x_2||_{H^2(\mathbb{R})}, \end{aligned}$$

$$(9.14)\qquad ||\bar{y}_1^4 - \bar{y}_2^4||_{H^2(\mathbb{R})} \leq C(\epsilon)||y_1 - y_2||_{L^\infty(\mathbb{R})} + C(\epsilon)||x_1 - x_2||_{H^2(\mathbb{R})}.$$

We consider the norm in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$

$$|||(x, y)||| = ||x||_{H^2(\mathbb{R})} + ||y||_{L^\infty(\mathbb{R})} + ||y||_{L^2(\mathbb{R})}.$$

Then provided $\epsilon > 0$ is sufficiently small, (9.13) yields

$$(9.15)\qquad |||\mathbf{T}^4(x_1, y_1) - \mathbf{T}^4(x_2, y_2)||| \leq \frac{1}{2}|||(x_1, y_1) - (x_2, y_2)|||, \quad \forall (x_i, y_i) \in \mathbf{B}_\epsilon.$$

Hence the map $\mathbf{T}^4 : \mathbf{B}_\epsilon \to \mathbf{B}_\epsilon$ is a contraction with respect to the norm $|||\cdot|||$. We define a sequence $u_n = (x_n, y_n) \in \mathbf{B}_\epsilon$ by $u_{n+1} = \mathbf{T}^4 u_n,\ n \geq 0,\ u_0 = (0, 0)$. Then $\{u_n\}$ is Cauchy with respect to the norm $|||\cdot|||$. Thus there exist $x_* \in H^2(\mathbb{R}),\ y_* \in L^\infty(\mathbb{R}),\ \tilde{y} \in L^2(\mathbb{R})$ such that

$$||x_n - x_*||_{H^2(\mathbb{R})} \to 0, \quad ||y_n - y_*||_{L^\infty(\mathbb{R})} \to 0, \quad ||y_n - \tilde{y}||_{L^2(\mathbb{R})} \to 0, \quad n \to \infty.$$

Relation (9.14) (noting the difference in notation) yields that for $n, m \geq 1$,

$$||y_{n+m} - y_n||_{H^2(\mathbb{R})} \leq C(\epsilon)||y_{n+m-1} - y_{n-1}||_{L^\infty(\mathbb{R})} + C(\epsilon)||x_{n+m-1} - x_{n-1}||_{H^2(\mathbb{R})}.$$

Thus the sequence y_n is Cauchy in $H^2(\mathbb{R})$ and $||y_n - y_*||_{H^2(\mathbb{R})} \overset{n\to\infty}{\to} 0$. Let $u_* = (x_*, y_*) \in \mathbf{B}_\epsilon$, then $u_n \overset{H^2 \times H^2}{\to} u_*$ as $n \to \infty$. From (9.13), (9.14) we have that $\mathbf{T}^4 u_n \overset{H^2 \times H^2}{\to} \mathbf{T}^4 u_*$ as $n \to \infty$. Since $u_{n+1} = \mathbf{T}^4 u_n$, it follows that $\mathbf{T}^4 u_* = u_*$. From (9.15) we get that u_* is the unique fixed point of $\mathbf{T}^4$ in $\mathbf{B}_\epsilon$. Note that

$\mathbf{T}^5 u_* = \mathbf{T}u_*$, or equivalently $\mathbf{T}^4(\mathbf{T}u_*) = \mathbf{T}u_*$. This implies that $\mathbf{T}u_* \in \mathbf{B}_\epsilon$ is also a fixed point of $\mathbf{T}^4$. Consequently $\mathbf{T}u_* = u_*$ which proves the Lemma. □

Theorem 9.1. *If $\epsilon > 0$ is sufficiently small, then there exists a solution $(x_\epsilon, y_\epsilon) \in C^2(\mathbb{R}) \times C^2(\mathbb{R})$ of (2.1), (2.2), (2.3) with*

$$||x_\epsilon - x_0||_{H^2(\mathbb{R})} + \epsilon^{\frac{1}{3}}||y_\epsilon - y_0||_{L^2(\mathbb{R})} + \epsilon^{\frac{2}{3}}||y_\epsilon - y_0||_{L^\infty(\mathbb{R})} \le C\epsilon.$$

Proof. From Lemma 9.2 and (9.4), (9.5) we have that $x_\epsilon := x_0 + x_*^\epsilon,\ y_\epsilon := Y_r^\epsilon + y_*^\epsilon$ satisfy with $b_\epsilon \in \mathbb{R}$:

$$\begin{aligned} x_\epsilon'' &= g_x(x_\epsilon, y_\epsilon) - b_\epsilon x_0' \\ \epsilon^2 y_\epsilon'' &= g_y(x_\epsilon, y_\epsilon) \end{aligned} \tag{9.16}$$

and (2.2), (2.3). Hence $x_\epsilon,\ y_\epsilon \in C^2(\mathbb{R})$ and

$$\frac{d}{ds}\left(\frac{1}{2}{x_\epsilon'}^2 + \frac{1}{2}\epsilon^2{y_\epsilon'}^2 - g(x_\epsilon, y_\epsilon)\right) = -b_\epsilon x_0' x_\epsilon', \quad s \in \mathbb{R}.$$

Integrating and using (2.2), (2.3), $x_\epsilon',\ y_\epsilon' \overset{|s|\to\infty}{\to} 0$ and $g(0,0) = g(1, \sqrt{2})$, we obtain

$$-b_\epsilon \int_{-\infty}^{\infty} x_0' x_\epsilon' ds = 0. \tag{9.17}$$

Note that

$$\int_{-\infty}^{\infty} x_0' x_\epsilon' ds = \int_{-\infty}^{\infty} x_0'(x_0' + x_*')ds \overset{x_* \in \mathbf{B}_\epsilon}{=} \int_{-\infty}^{\infty} x_0'^2 ds + o_\epsilon(1) > 0$$

provided $\epsilon > 0$ is sufficiently small. Thus (9.17) yields $b_\epsilon = 0$. Consequently via (9.16), (2.2), (2.3) we have proved the existence part of the Theorem. The estimate of the Theorem follows from the fact that $(x_*, y_*) \in \mathbf{B}_\epsilon$ and (6.10), (6.11). The proof of the Theorem is complete. □

Appendix A. Proof of the technical estimates of Proposition 4.1

Proof of (4.11)

<u>Claim 1.</u> There exists a sequence $\{s_n\}_{n\ge1}$ such that $s_n \to \infty$ as $n \to \infty$ and

$$R'(s_n) \le \sqrt{\frac{bm}{a}} {s_n}^{-\frac{1}{2}}, \quad n \ge 1. \tag{A.1}$$

Indeed, suppose that the claim is not true. Then there exists $C > 0$ such that

$$R'(s) \ge \sqrt{\frac{bm}{a}} s^{-\frac{1}{2}}, \quad s \ge C.$$

Integrating the above, we get

$$R(s) \ge 2\sqrt{\frac{bm}{a}s} - C', \quad s \ge C,$$

which contradicts (4.9). This proves Claim 1.

We set $q(s) := 2aR\left(R + \sqrt{\frac{bm}{a}s}\right)$, $s > 0$. Then from (4.9) we have

$$3bms \le q(s) \le 5bms \quad \text{for large } s. \tag{A.2}$$

Also from (4.6) :

$$R'' = q(s)\left(R - \sqrt{\frac{bm}{a}s}\right), \quad s > 0. \tag{A.3}$$

<u>Claim 2.</u> There exists a sequence $\tilde{s}_n$ such that $\tilde{s}_n \to \infty$ as $n \to \infty$ and

$$\left|R(\tilde{s}_n) - \sqrt{\frac{bm}{a}\tilde{s}_n}\right| \leq \frac{1}{\sqrt{bma}}\tilde{s}_n^{-\frac{5}{2}}. \tag{A.4}$$

We argue by contradiction. Suppose that there exists $C > 0$ such that

$$\left|R(s) - \sqrt{\frac{bm}{a}s}\right| \geq \frac{1}{\sqrt{bma}}s^{-\frac{5}{2}}, \quad s \geq C. \tag{A.5}$$

We may assume that

$$q(s) \geq 3bms, \quad s \geq C \quad \text{(from (A.2))}, \tag{A.6}$$

and

$$R'(C) \leq \sqrt{\frac{bm}{a}}C^{-\frac{1}{2}} \quad \text{(from (A.1))}. \tag{A.7}$$

We consider two cases:

(i) $R(s) - \sqrt{\frac{bm}{a}s} \leq -\frac{1}{\sqrt{bma}}s^{-\frac{5}{2}}, \quad s \geq C.$

Then from (A.3), (A.6)

$$R''(s) \leq (3bms)\left(-\frac{1}{\sqrt{bma}}s^{-\frac{5}{2}}\right) = -3\sqrt{\frac{bm}{a}}s^{-\frac{3}{2}}, \quad s \geq C.$$

Integration yields,

$$R'(s) \leq 6\sqrt{\frac{bm}{a}}s^{-\frac{1}{2}} - 6\sqrt{\frac{bm}{a}}C^{-\frac{1}{2}} + R'(C), \quad s \geq C.$$

From the above relation and (A.7) we get $R'(s) < 0$ for large s which contradicts the fact that $R'(s) > 0$, $s \in \mathbb{R}$.

(ii) $R(s) - \sqrt{\frac{bm}{a}s} \geq \frac{1}{\sqrt{bma}}s^{-\frac{5}{2}}, \quad s \geq C.$

This case also leads to a contradiction by similar more simple arguments.
Thus both cases give a contradiction, which proves Claim 2.

Let $u(s) := R(s) - \sqrt{\frac{bm}{a}s}$, for $s > 0$. Then from (A.3) we obtain that for $s > 0$:

$$-u'' + q(s)u + \frac{1}{4}\sqrt{\frac{bm}{a}}s^{-\frac{3}{2}} = 0. \tag{A.8}$$

<u>Claim 3.</u>There exists a large $C > 0$ such that

$$\underline{u}(s) = -\frac{1}{\sqrt{bma}}s^{-\frac{5}{2}}, \quad \bar{u}(s) = \frac{1}{\sqrt{bma}}s^{-\frac{5}{2}}$$

are sub-supersolutions respectively of (A.8) in $[C,\infty)$.
Indeed,

$$\begin{aligned}
-\underline{u}''+q(s)\underline{u}+\frac{1}{4}\sqrt{\frac{bm}{a}}s^{-\frac{3}{2}} &= \frac{35}{4}\frac{1}{\sqrt{bma}}s^{-\frac{9}{2}}-q(s)\frac{1}{\sqrt{bma}}s^{-\frac{5}{2}}+\frac{1}{4}\sqrt{\frac{bm}{a}}s^{-\frac{3}{2}}\leq \\
&\overset{(A.2)}{\leq} \frac{35}{4}\frac{1}{\sqrt{bma}}s^{-\frac{9}{2}}-(3bms)\frac{1}{\sqrt{bma}}s^{-\frac{5}{2}}+\frac{1}{4}\sqrt{\frac{bm}{a}}s^{-\frac{3}{2}}= \\
&= \frac{35}{4}\frac{1}{\sqrt{bma}}s^{-\frac{9}{2}}-3\sqrt{\frac{bm}{a}}s^{-\frac{3}{2}}+\frac{1}{4}\sqrt{\frac{bm}{a}}s^{-\frac{3}{2}}= \\
&= s^{-\frac{9}{2}}\left(\frac{35}{4}\frac{1}{\sqrt{bma}}-\frac{11}{4}\sqrt{\frac{bm}{a}}s^{3}\right)<0
\end{aligned}$$

if $s\geq C$ large enough. Thus $\underline{u}$ is a subsolution of (A.8) in $[C,\infty)$.
The fact that $\bar{u}$ is a supersolution of (A.8) in $[C,\infty)$ follows from the previous relation and from $\bar{u}=-\underline{u}$.

Based on Claim 2, we can assume that $u,\underline{u},\bar{u}$ are solutions, subsolutions, supersolutions respectively of (A.8) in $[C,\infty)$ with $|u(C)|\leq\frac{1}{\sqrt{bma}}C^{-\frac{5}{2}}$. We thus have $\underline{u}(C)\leq u(C)\leq\bar{u}(C)$ and from (4.9) that $u,\underline{u},\bar{u}\to 0$ as $s\to\infty$. By combining the above with $q(s)>1$, $s\geq C$, it follows from the maximum principle that $\underline{u}(s)\leq u(s)\leq\bar{u}(s)$, $s\geq C$ i.e.

$$\left|R(s)-\sqrt{\frac{bm}{a}}s\right|\leq\frac{1}{\sqrt{bma}}s^{-\frac{5}{2}}\quad,\quad s\geq C.$$

This proves (4.11).
Proof of (4.12)
From (4.6), (4.9) we have

$$|R''(s)|\leq Cs^{-\frac{3}{2}},\quad s>0. \tag{A.9}$$

Since $R'(s)>0,\ s\in\mathbb{R}$, Claim 1 implies that $R'(s_n)\overset{n\to\infty}{\to}0$. Thus

$$R'(s)-R'(s_n)=\int_{s_n}^{s}R''(t)dt\ \overset{(A.9)}{\Rightarrow}\ R'(s)=\int_{\infty}^{s}R''(t)dt,\ s\in\mathbb{R},$$

which via (A.9) gives (4.11) for $s>0$. For $s<0$ the proof is similar.

Appendix B. Proof of Proposition 5.1

Proof. Note that (4.10), (4.11) yield

$$3aR^2(s)-bms\to\infty,\quad\text{as}\quad|s|\to\infty. \tag{B.1}$$

This implies that y has finitely many zeros in $\mathbb{R}$ (all of them simple). Using this, the differential equation, and $y\in L^{\infty}(\mathbb{R})$, we get $y,y',y''\to 0$ exponentially as $|s|\to\infty$.

Suppose that y is not identically zero.
(i) If $y(s)\neq 0,\ \forall s\in\mathbb{R}$. Then multiplying (5.7) with $R'>0$ and integrating by

parts gives

$$\mu \int_{-\infty}^{\infty} yR' ds = \int_{-\infty}^{\infty} [-R''' + 2(3aR^2(s) - bms)R'] y ds \overset{(4.6)}{=} \int_{-\infty}^{\infty} 2bmRy ds.$$

Thus $\mu > 0$, a contradiction. Integration by parts is justified from the previous observation and elementary bounds on R', R'', R''' which follow from Proposition 4.1.
(ii) If $y(k) = 0$ for some $k \in \mathbb{R}$. Then we may assume without loss of generality that $y'(k) > 0$ and $y(s) > 0$, $s > k$. Again multiplying (5.7) with R' but this time integrating by parts over $[k, \infty)$ gives $\mu > 0$, a contradiction.
The Proposition is proved. □

Appendix C. Proof of Lemma 5.2

Proof. Suppose that the claim of the Lemma is not true. Then there exist $K_n \overset{n\to\infty}{\to} \infty$, $\lambda_n \le 0$, $y_n \in C^2[-K_n, K_n]$, $n \ge 1$, such that

$$-y_n'' + 2(3aR^2(s) - bms)y_n = \lambda_n y_n, \qquad s \in [-K_n, K_n], \tag{C.1}$$

$$y_n'(-K_n) = y_n'(K_n) = 0, \qquad ||y_n||_{L^\infty[-K_n, K_n]} = 1. \tag{C.2}$$

From (C.2) we may assume without loss of generality that there exist $s_n \in [-K_n, K_n]$, $n \ge 1$ such that

$$y_n(s_n) = 1 \ , \quad y_n'(s_n) = 0 \ , \quad y_n''(s_n) \le 0.$$

Setting $s = s_n$ in (C.1) gives

$$2(3aR^2(s_n) - bms_n) \le \lambda_n \le 0 \ , \qquad n \ge 1, \tag{C.3}$$

i.e.

$$\min_{s\in\mathbb{R}} 2(3aR^2(s) - bms) \le \lambda_n \le 0 \ , \qquad n \ge 1, \quad \text{(cf. (B.1))}. \tag{C.4}$$

From (C.1), (C.2), (C.4) and the usual diagonal argument (passing to a subsequence):

$$y_n \to \bar{y} \quad \text{in} \quad C^2_{loc}(\mathbb{R}) \ \text{ with } \ ||\bar{y}||_{L^\infty(\mathbb{R})} \le 1 \ , \quad \lambda_n \to \bar{\lambda} \le 0 \ \ , \ n \to \infty,$$

and

$$-\bar{y}'' + 2(3aR^2(s) - bms)\bar{y} = \bar{\lambda}\bar{y}, \qquad s \in \mathbb{R} \ .$$

To arrive at a contradiction, we show that $\bar{y}$ is not identically zero (cf. Proposition 5.1). Relations (C.3), (B.1) yield that the sequence $\{s_n\}$ is bounded. Thus, by passing to a subsequence, we may assume that $s_n \to \bar{s}$, as $n \to \infty$. Since $y_n \to \bar{y}$ in $C^2_{loc}(\mathbb{R})$ as $n \to \infty$ and $y_n(s_n) = 1$, we obtain $\bar{y}(\bar{s}) = 1$. The proof is complete. □

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