Overview of a Methodology for Scaling the Indeterminate Equations of Wall Turbulence

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Recent efforts by the present authors have focused on the fundamental multiscaling behaviors of the time averaged dynamical equations of wall-turbulence. These efforts have generated a number of new results relating to dynamical structure, as well as a new mathematical foundation. Central to this has been the development of the so-called method of scaling patches. This method provides a formalism for determining the scaling behaviors directly from the indeterminate equations. A general description of this methodology is provided herein, and in doing so its connections to well-established scaling notions are identified. Example problems for which the method has been successfully applied includes turbulent boundary layer, pipe and channel flows, turbulent Couette-Poiseuille flow, and fully developed turbulent heat transfer in a channel.

Nomenclature

T	Reynolds shear stress, $T = -\rho\overline{u'v'}$
u	Velocity component in the x direction
u_τ	Friction velocity, $u_\tau = \sqrt{\tau_w/\rho}$
v	Velocity component in the y direction
U	Mean axial velocity component
U_∞	Freestream velocity
x	Streamwise coordinate direction
y	Wall-normal coordinate direction
β	Scale hierarchy parameter
δ	Boundary layer thickness, channel half-height
η	Outer normalized distance from the wall, $\eta = y/\delta$
ϵ	Small parameter, $\epsilon^2 = 1/\delta^+$
τ_w	Mean wall shear stress
ν	Kinematic viscosity
ρ	Mass density
Superscript	
m	Denotes maximum value
Superscript	
$+$	Denotes inner normalization
$*$	Denotes a general normalization
$'$	Denotes fluctuation about the mean

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I. Introduction

The present effort considers the scaling behaviors of the time averaged differential equations associated with wall bounded turbulent flows. Like their laminar counterparts, such equations, along with their boundary conditions, pose a singular perturbation type problem. The primary complication in scaling such equations, however, lies in the fact that they are underdetermined. This fact arises owing to the Reynolds stress gradient term that appears as a result of time averaging. The effect of time averaging is, of course, at the heart of the turbulence closure problem, and, with regard to scaling, it is rational to expect this averaging process to obscure and limit what might be derivable from analysis of the governing equations alone.

Given this context, on-going studies have been devoted toward elucidating what the time averaged equations themselves admit with regard to scaling, as well as the associated implications pertaining to flow physics.[1–4, 9] In this regard, the objectives of the present paper are to: *i*) provide an exposition of the rationale for, and overview of, the underlying methodology (the method of *scaling patches*) that has emerged from these efforts, and *ii*) provide an accounting of some of the scaling behaviors that have been discovered and new flow physics that have been revealed.

A. The Empirical Test for Scaling and its Minimal Mathematical Requirement

Experimentally based studies exploring, for example, Reynolds number dependence employ a well-established procedure for determining the "appropriate" normalization for statistical profiles across the flow. Given this, a recapitulation and examination of the implications of this empirical test for determining the validity of any given normalization (existence/non-existence of a particular scaling) provides a useful starting point.

Consider the typical case in which wall-normal profiles of a velocity field statistic, say $\phi(y)$, are acquired over a range of Reynolds numbers. In their dimensional form, these measured statistical profiles (empirically determined functions) will generally vary widely in their magnitude and shape as the Reynolds number is varied. For each point in the profile, the statistic and the y value are made non-dimensional according to the normalization being tested (e.g., inner or outer[a]). If the different Reynolds number profiles (or more likely a portion of the profiles) merge to a single curve under the given normalization, then the scaling is said to be appropriate (successful) over the indicated sub-domain. This is shown in Figs. 1a,b which compare dimensional and inner normalized mean velocity profile data for a zero pressure gradient boundary layer.

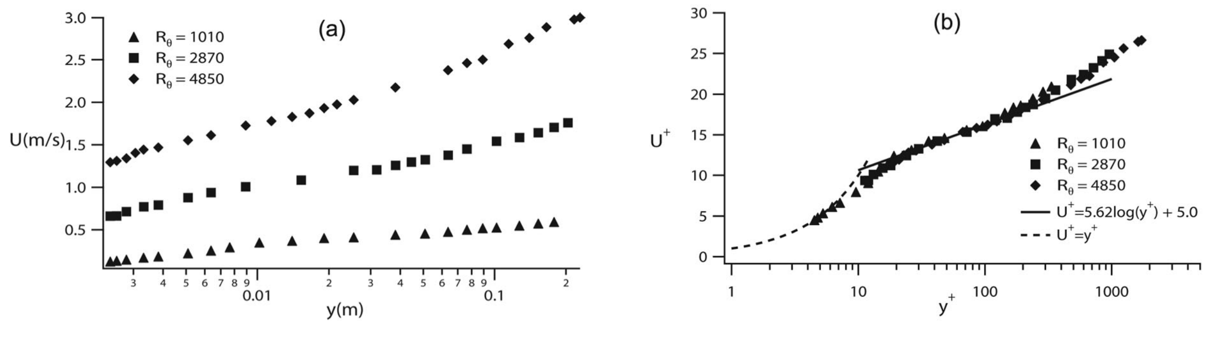

Figure 1. Dimensional (a) and inner normalized (b) mean velocity profiles in turbulent boundary layers.

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[a] The conventional inner normalization involves the use of the kinematic viscosity, ν, and the friction velocity, $u_\tau = \sqrt{\tau_w/\rho}$, where τ_w is the mean wall shear stress and ρ is the mass density. Similarly, outer normalization makes use of the external freestream velocity, U_∞ (or perhaps the channel centerline velocity), and some measure of the overall thickness of the flow like δ the boundary layer thickness.

Operationally, it is thus apparent that the successful normalization must stretch or compress the statistical function and its y value such that the normalized function, $\phi^*(y^*)$, is rendered invariant with changes in Reynolds number. For example, Fig. 1b explicitly shows that under inner normalization the distribution of the mean velocity, $U(y)$, is invariant with Reynolds number over a portion of the flow domain near the wall. Thus, at a minimum, attaining such a situation requires that both the normalized values of ϕ and the normalized values of y individually retain the same order of magnitude as the Reynolds number is varied. This said, it is relevant to note that satisfying this criterion does not necessarily guarantee that the normalized profiles will merge to a single invariant curve. If this criterion is not satisfied, however, it is incontrovertibly true that the profiles will fail to merge.

The scaling patch methodology described herein inherently invokes these requirements by recognizing that the desired statistical profile can be identified as a solution to an averaged transport equation, and thus its scaling behaviors can be determined through an analysis of this equation. This is done by *i*) requiring that the normalized forms of the indeterminate equations maintain consistency with this minimal criterion, and *ii*) simultaneously identifying those normalizations that appropriately retain the dominant terms in the governing equations. In the general application of the method, physical arguments supported by known properties and/or empirical data are used to identify the dominant terms of the equations. This, of course, is a well-established procedure (see the laminar boundary layer example below), although arguably has been under-utilized in addressing turbulent flow scaling problems.

II. Formalization of the Scaling Framework

Operationally, the above described criteria, as applied toward identifying a successful scaling, requires that the governing equation recovers a normalized form that is explicitly parameter free. This requirement stems from the fact that the normalization that properly scales the equation must implicitly yield the desired stretching/compressing attributes mentioned above (i.e., yields a transformation of the equations that is, for example, invariant with Reynolds number). The mathematical formalism employed to identify such normalizations includes the search for *scaling patches* through the use of *admissible differential scalings*.[4] These are now described.

Succinctly, a scaling patch is a y-subdomain over which the normalized variables in the governing equation remain $O(1)$ (bounded) as $R \to \infty$ and that at least one scaled derivative of the dependent variable is $O(1)$, with the others being no larger. Admissible differential scalings are those that recover the desired $O(1)$ variations in the scaled variables over the scaling patch while retaining a form of the governing equation that has at least two terms of dominant order of magnitude.

Given these definitions it it first important to note that the basic averaged physical law of fluid dynamics, upon which all the considerations presented below are grounded, is the mean momentum balance (MMB) equation obtained by Reynolds averaging the Navier Stokes equations. Secondly, the notion of scaling is a linear, parameter dependent transformation of independent and dependent variables, which of course results in a different form for the MMB. A scaling patch,[3,4] employs a scaling, together with an interval of distances from the wall in which the scaling under consideration is natural for the flow. Relevant to turbulent wall-flows, this concept has similarities to Prandtl's and von Karman's original phenomenological notion of mixing lengths[5] and their domains of validity. That means that the derivatives of the scaled variables with respect to the scaled distance are $O(1)$, i.e. numerically bounded independent of the problem's small or large parameters. Examples, of course, are the traditional inner and outer domains. Automatically, the width of a scaling patch is at least as large as the characteristic length associated with the scaling.

A central aspect of the approach is based on a systematic method for locating scaling patches. Inherent to the method are the following, rational yet assumed, criteria for the determination of a scaling patch:

- The proposed scaling must transform the MMB equation into an equation that still expresses a balance between force-like quantities (which, of course, is what the original MMB equation does).
- The proposed patch must be compatible with the flow, in that it can be shown rigorously by other

means that certain actual derivatives of the flow quantities at a position in the patch are of the order of magnitude implied by the scaling.

The procedure then allows the systematic identification of the scaling patches using this criterion. For the wall-bounded turbulent flows considered thus far, there is a continuum of them parameterized by a parameter, β, which can with straightforward analysis be correlated with, for example, positions across the channel. This correlation is not written down explicitly, except in order of magnitude. Thus in an interval of distances from the wall, that stretches almost all the way across the flow, each location is the seat of a scaling patch within which the proper scaling of variables is known. Asymptotically as $\beta \rightarrow 0$, it is shown that the characteristic length in a given patch, which coincides in order of magnitude with the patch's width, is also proportional to its distance from the wall. This property is also reminiscent of mixing lengths. With an additional reasonable assumption, it can also be shown that in any interior subinterval of the hierarchy defined in a definite way, the mean velocity profile approaches a logarithmic one as the Reynolds number approaches infinity.[3]

Examples relating to both laminar and turbulent flows are provided below. In the laminar flow example, the above criteria are shown to lead to the identification of a parameter-free normalization and subsequently recover the well-known similarity scaling. In this instance, the entire boundary layer can be viewed as encompassing a single scaling patch. In the turbulent case, the traditional inner and outer normalizations are shown to be appropriate over scaling patches that respectively cover only a portion of the flow. In either example, identification of the scaling patch requires invoking known physical behaviors or the examination of data in order to verify that the dominant terms in the governing equation are appropriately recovered. An essential aspect of the analysis involves use of the unintegrated form of the governing equation.

III. Results

Scaling patches for a number of flows are now identified, and in doing so, the associated mathematical and physical structure of the flow is revealed.

A. Laminar Boundary Layer Flow

The properties of the laminar boundary layer are established through the simultaneous action of advection and diffusion. As is well known (and derivable via a number of methods), these dynamical mechanisms underlie the classical $R^{1/2}$ scaling behavior. In the following, a well-known derivation for this scaling is provided that exemplifies elements that are also generic to the scaling patch approach.

Consider steady, laminar, two-dimensional, incompressible boundary layer flow in the x direction over a flat no-slip surface located at $y = 0$. Under these conditions, the momentum and continuity equations reduce to:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

The goal now is to find a normalization that appropriately captures the dynamical balance between advection and diffusion at all Reynolds numbers.[b] Such a normalization, of course, is one that renders (1) parameter free (i.e., the Reynolds number dependence becomes embedded within the normalization). With this aim in mind we define $u^* = u/U_\infty$ and $x^* = x/L$, where U_∞ is the freestream velocity and L is the distance down the plate. To allow for the possibility of a parameter free normalization of (1), normalizations for v

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[b] Note that this balance must exist everywhere in the flow since, even after the limiting process by which the boundary layer approximations are determined, (1) still contains two types of terms.

and y are sought using the Reynolds number dependent scalings, $v^* = R^n v/U_\infty$ and $y^* = R^m y/L$, where $R = U_\infty L/\nu$.

Normalization of (2) results in,

$$\frac{1}{R^{m-n}}\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{3}$$

and normalization of (1) yields,

$$u^*\frac{\partial u^*}{\partial x^*} + \frac{R^n}{R^m}v^*\frac{\partial u^*}{\partial y^*} = R^{2m-1}\frac{\partial^2 u^*}{\partial y^{*2}} \tag{4}$$

Because this system of equations is closed, to determine the values of m and n one may employ purely analytical methods.[6] In connection with the method of scaling patches and the indeterminate equations of wall turbulence, however, it is more relevant to illustrate how physical arguments that are, for example, based on empirical observations can be effectively employed to determine the underlying scaling behaviors.

Since the boundary layer approximations have already been invoked, the two terms in (3) must always be equal and opposite. This requires that $m = n$. Relative to (4) the following observations are made.

1. If $m > 1/2$, then as $R \to \infty$ the equation reduces to $\partial^2 u^*/\partial y^{*2} \to 0$, or $\partial u^*/\partial y^* \to const.$
2. If $m < 1/2$, then as $R \to \infty$ one $\partial^2 u^*/\partial y^{*2}$ becomes negligibly small relative to the terms on the left.

The condition $m > 1/2$ demands that as the Reynolds number becomes large the shear stress approaches a constant. Physically, this possibility is rejected since it is contrary to the central notion that a boundary layer is a zone of non-zero vorticity; thus demanding the existence of a vorticity gradient across the layer. Furthermore, and perhaps at an even more fundamental level, the $m > 1/2$ case also leads to a condition in which only one term is left in the momentum equation. This is physically highly questionable since the co-dominant terms in the boundary layer were already established in the derivation of the boundary layer equation. Similarly, the $m < 1/2$ case leads to an inviscid equation for the flow within the laminar boundary layer at high Reynolds number. This conclusion can also be summarily rejected since it is contrary to the irreducibility of viscous effects in the boundary layer. From these considerations it is thus concluded that $m = n = 1/2$. Under this condition the governing equations become parameter free,

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{5}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{\partial^2 u^*}{\partial y^{*2}}, \tag{6}$$

and the classical scalings, $v^* = R^{1/2}v/U_\infty$, $y^* = R^{1/2}y/L$ are recovered.

Important generic elements of the above analysis include, *i*) the search for a parameter free form of the momentum balance that continues to reflect the dominant dynamical mechanisms, and *ii*) the use of reasonable physical arguments (supported by empirical evidence) to guide the analysis. These rather classical elements are also central to the scaling patch methodology as applied the under-determined equations of wall turbulence. As shown, in the laminar flow case the entire flow is covered by a single scaling patch, while in turbulent flows multiple scaling patches may exist.

B. A Prototypical Indeterminate Case: Turbulent Couette Flow

Elements of the scaling patch methodology as applied to a prototypical case of turbulent Couette flow are now provided. This is turbulent flow through a channel in which the forcing is provided not by a pressure gradient, but rather by the motion of the upper wall relative to the lower wall. Only a brief outline is provided here; a much more complete exposition can be found in the studies by Fife et al.[3,4]

The averaged equation of streamwise momentum balance for steady turbulent Couette flow expresses an exact balance between the transverse gradients of the viscous and Reynolds stresses. In inner (wall) dimensionless variables, this equation is

$$\frac{d^2U^+}{dy^{+2}} + \frac{dT^+}{dy^+} = 0, \tag{7}$$

where T^+ is the inner normalized Reynolds shear stress. There are also boundary conditions for the two unknowns U^+ and T^+. Physically, this equation has a level of analogy with the laminar case just considered. That is, the momentum equation in the laminar case effectively describes a balance between axial advection and wall normal diffusion. Similarly, the mean momentum equation in turbulent Couette flow describes the average balance between wall normal turbulent advection and wall normal diffusion.

The analysis proceeds with small parameter $\epsilon^2 = {\delta^+}^{-1}$, where δ^+ is a Reynolds number based on the friction velocity. The outer length variable is $\eta = \epsilon^2 y^+$. The channel centerline is at $\eta = 1$. If we define a scaled Reynolds stress $\hat{T}(\eta)$ by

$$T^+ = T^+_m + \epsilon^2 \hat{T}, \tag{8}$$

where $T^+_m = T^+_{\eta=1}$, then the outer-scaled averaged momentum equation is seen to be

$$\frac{d^2U^+}{d\eta^2} + \frac{d\hat{T}}{d\eta} = 0. \tag{9}$$

These are just two scaling patches. A continuous family of them can be found by use of a family of "adjusted Reynolds stresses"

$$T^\beta(y^+) = T^+(y^+) - \beta y^+, \tag{10}$$

where β is a parameter with a certain range which can be found from the properties of the function $P(y^+) = \frac{dT^+}{dy^+}(y^+)$. Specifically, since P vanishes at the wall and the centerline, and T^+ assumes positive values, then so does P, and there is guaranteed to exist an interval where P decreases from a local (maybe global) maximum to 0. The range of the function P in that interval is very nearly the range of values of β. The actual numbers delineating this range can be found from empirical data.

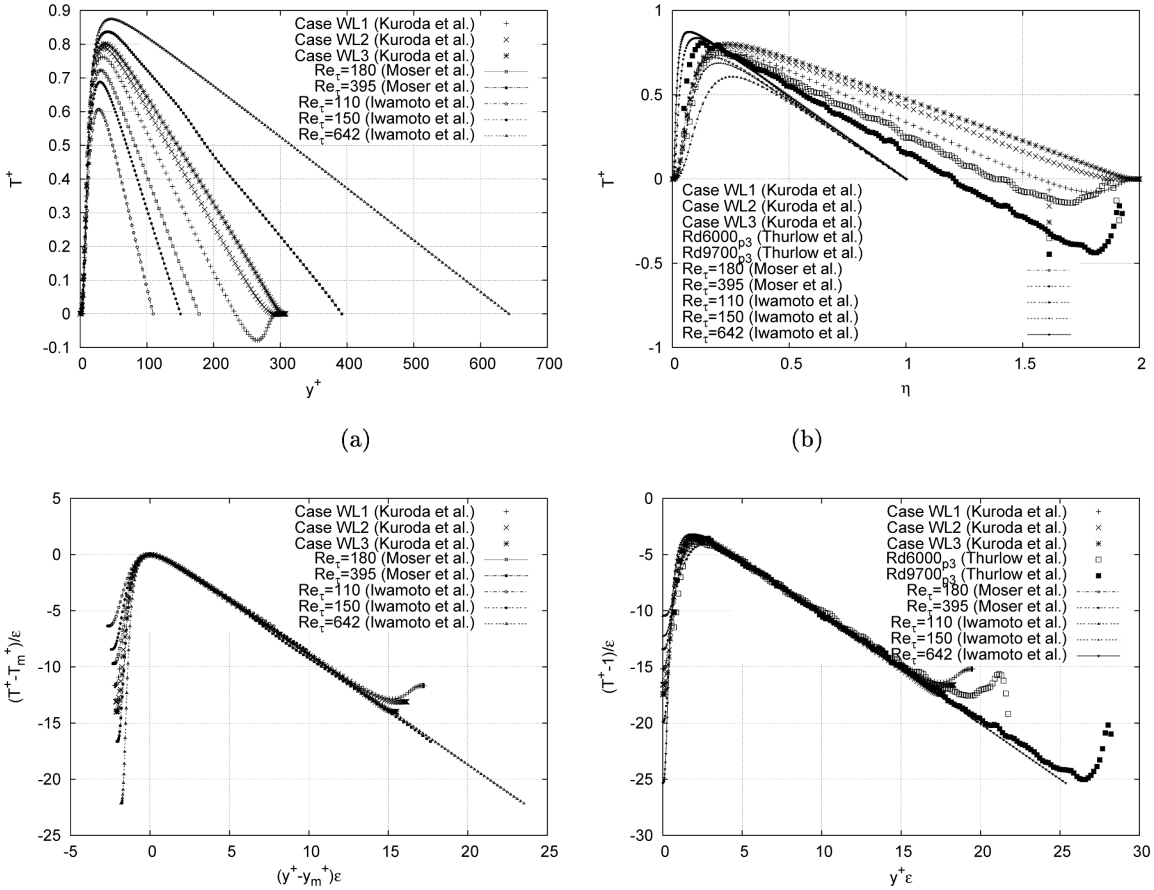
This is also the set of values of β for which the graph of $T^\beta(y^+)$ has a local maximum at some location $y^+ = y^\beta_m$. It can readily be shown using the scaling patch criteria listed above that there is a scaling patch located at that maximum, and the characteristic length in that patch is $O(\beta^{-1/2})$. The patch with the least value of β, namely $\beta = \epsilon^4$, is essentially identical to the outer scaling domain.

Further analysis allows one to correlate values of β with locations in the channel, and to determine many qualitative properties of the profiles $U^+(y^+)$, $T^+(y^+)$. For example, important information about the classical question of the existence of logarithmic-like profiles for U^+ is obtained. Also, the sense in which the characteristic length is or is not proportional to distance from the wall, i.e. to y^β_m, is clarified.[3]

C. Extended Examples

Pure Couette flow is solely composed of a layer who dynamical balance is between the mean viscous force and mean turbulent inertia. Wei et al.[1] identify this as a *stress gradient balance layer*, and show that such layers also exist in boundary layer, pipe and channel flows. They further show that in channel flow, for example, there is also an authentic outer layer in the sense that the dominant terms in the momentum balance are the Reynolds stress gradient and mean axial pressure gradient (i.e., are inviscid mechanisms). In connection with this, they show that in moving from the stress gradient balance layer to the outer layer a mathematical *balance breaking and exchange* process occurs. This process is identical in structure to that which occurs between the different layers in the hierarchy described above. In the case of pipe and channel flows, Wei et al.[2] effectively use the properties of intermediate scaling patch that exists between the stress gradient balance layer and the outer layer (layers II and IV in their nomenclature) to scale (meso-scale) the Reynolds stress in the interior zone (of size $O(\delta^{+1/2})$) where both inner and outer normalizations fail to merge the differing Reynolds number data.

A common element of all these cases is, however, that the scaling is solely parameterized by the Reynolds number. Efforts to extend the scaling patch method to more complicated cases are on-going. Specifically, the turbulent Couette-Poiseuille flow and channel flow heat transfer problems discussed below are characterized by parameters representing multiple physical effects. In Couette-Poiseuille flow the additional effect is the imbalance of the shear stesses at the upper and lower walls owing to the relative wall motion. Similarly, in fully developed turbulent heat transfer in a channel there are the similtanueous effects of Prandtl and Reynolds number. In what follows, these extentions to the method are briefly discussed.



(a)

(b)

(c)

(d)

Figure 2. Normalizations of the Reynolds shear stress in turbulent Couette-Poiseuille flow for δ^+ ranging between 110 and 642 and shear stress ratios ranging between 0.5 and 1.0: (a) inner scaling, (b) outer scaling, (c) meso-scaling, (d) approximate meso-scaling.[7]

Tubulent Couette-Poiseuille Flow: This is the case of fully developed two-dimensional (in the mean), turbulent flow in a channel that is simultaneously driven by a mean axial pressure gradient and a relative wall motion in the plane of the wall. The mathematical formulation of this flow is aided by the fact that, owing to symmetry considerations, one can always formulate the problem such that the pressure gradient generating flow is in the positive x direction and the upper wall moving. Wei et al.[7] show that the inner normalized MMB for this flow can be expressed as,

$$\frac{d^2U^+}{dy^{+2}} + \frac{dT^+}{dy^+} + \epsilon^2\omega = 0, \tag{11}$$

where ϵ is defined as previously (using the friction velocity at the lower wall), and $0 \leq \omega \leq 1$, depending on the Couette and Poiseuille contributions. Relative to the balance breaking and exchange process that occurs in pure Poiseuille flow, the meso-scaling is modified owing to the appearance of ω. An analogous analysis can be constructed, however, and the results in Fig. 2 compare the traditional inner and outer scalings with the meso-scaling. As with Poiseuille flow alone,[2] an approximate meso-scaling is also possible.

Fully Developed Heat Transfer in a Channel: Wei et al.[8] consider fully developed turbulent heat transfer in a channel owing the presence of a constant wall heat flux. In this case the relevant transport equation is the thermal energy balance, which, when appropriately expressed, attains a form similar to that of the inner normalized MMB for turbulent Couette-Poiseuille flow,

$$\frac{d^2\Psi}{dy_\sigma^2} + \frac{dT_\theta}{dy_\sigma} + \sigma^2 r_\sigma = 0 \tag{12}$$

In this equation Ψ is the deviation from the wall temperature normalized by the maximum deviation, T_σ the normalized turbulent heat flux, σ is a Prandlt (Peclet) number dependent parameter associated with the maximum inner normalized temperature difference, y_σ is a modified inner normalized wall-normal distance, and r_σ is the Reynolds number dependent function expressing the ratio of the local and bulk velocities ($r_\sigma = O(1)$ over most of the channel). As in the case of Couette-Poiseuille flow the last term contains two components, in this case, however, the problem is mathematically richer since r_σ is a function of y_σ. A modified multiscale analysis remains available, however, and the results of Fig. 3 compare the traditional inner normalization with the resulting meso-scaling.

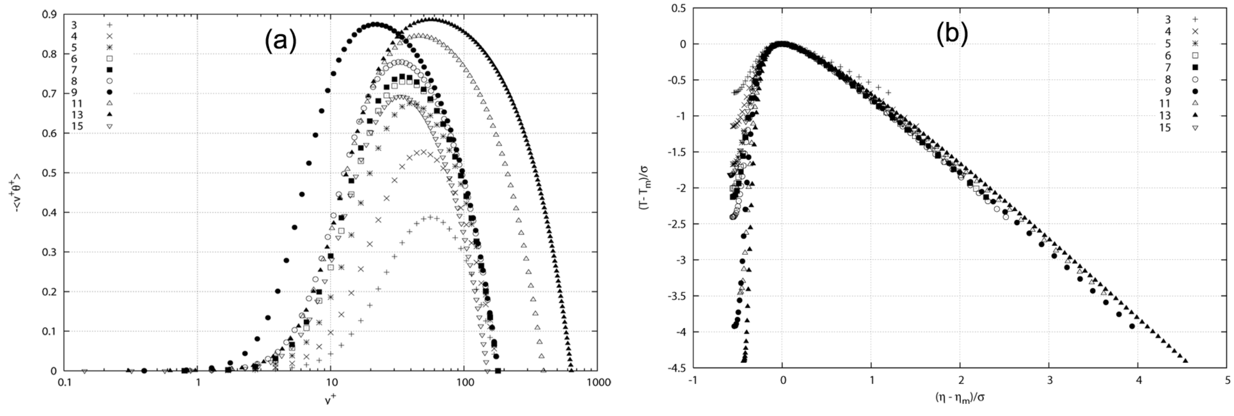


Figure 3. Normalization of the turbulent heat flux in fully developed channel flow for Prandtl numbers ranging from 0.025 to 5.0 and Reynolds numbers, δ^+, ranging from 150 to 642: (a) inner scaling, (b) meso-scaling.[8]

IV. Conclusion

This paper provides an overview of the method of scaling patches as applied to the under-determined equations of wall-turbulence. The method employs generic elements that are well-established relative to scaling the equations of fluid dynamics. Arguably, these notions have previously not been sufficiently formalized and thus have been under-utilized relative to determining the scaling behaviors of the Reynolds averaged Navier-Stokes (RANS) equations. As shown by the examples herein, the method can reveal the natural scalings over the associated domains (patches) directly from analysis of the RANS equations. By doing so, the method has also generated a picture of turbulent wall-flow physics that is a considerable departure from the predominant view.[1,3,9] On-going efforts continue to extend the methodology.

Acknowledgments

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