Quadratic programming

$Q \in \mathbb{R}^{n\times n}$ symm

$$\begin{cases} \min \frac{1}{2} x^T Q x + x^T d \\ \text{s.t. } a_i^T x = b_i & i \in E \\ \quad\;\; a_i^T x \geq b_i & i \in I \end{cases}$$

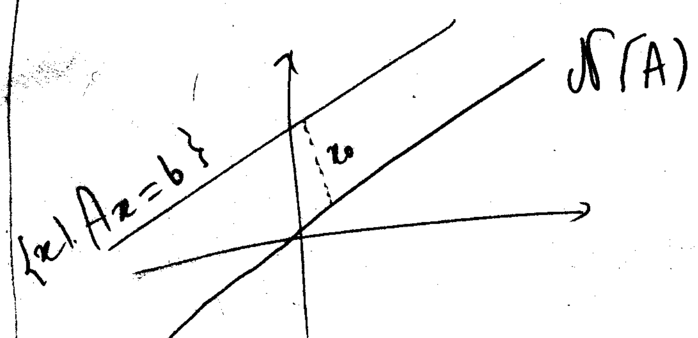
Equality constrained QP:

$$\begin{cases} \min \frac{1}{2} x^T Q x + x^T d \\ \text{s.t. } Ax = b \end{cases}$$

$Q \in \mathbb{R}^{n\times n}$ symm
$A \in \mathbb{R}^{m\times n}$, $\text{rank}(A) = m$, $m < n$

Note: assuming $rk(A) = m$ is not restrictive. Can always do preprocessing in A to throw away redundant rows (The corresponding b must be consistent otherwise the problem has no solution)

Nullspace $A = \mathcal{N}(A) = \{x \mid Ax = 0\}$



If we know $\mathcal{N}(A)$ and some point x_0 s.t. $Ax_0 = b$ then we know feasible set.

Characterization of $\mathcal{N}(A)$:

Would like to find $Z \in \mathbb{R}^{n \times (n-m)}$ s.t. columns of Z form a basis of $\mathcal{N}(A)$ i.e.:

$$\begin{cases} \text{rank}(Z) = n - m = \dim \mathcal{N}(A) \\ AZ = 0 \end{cases}$$

Here are two ways of doing this:

1) Suppose we have identified m lin. indep columns of A and assume w.l.o.g. that they are the first m columns:

$$A = (\underbrace{B}_{m\times m}\ \underbrace{N}_{n-m\times m}) \qquad (\ \text{range}(A) = \text{range}(B))$$

Take $Z = \begin{pmatrix} -B^{-1}N \\ I \end{pmatrix} \in \mathbb{R}^{n\times n-m}$; $\text{rank}(Z) = n-m$

$$AZ = (B\ N)\begin{pmatrix} -B^{-1}N \\ I \end{pmatrix} = 0$$

2) $A^T = QR$

$$= (\underbrace{Q_1}_{m} | \underbrace{Q_2}_{n-m}) \begin{pmatrix} \nabla \\ 0 \end{pmatrix} = (Q_1 \mid Q_2)\begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix}$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal and $\tilde{R} \in \mathbb{R}^{m\times m}$ is upper triangular

$$Q^T A^T = R \qquad \underset{(AQ_1 \mid AQ_2)}{\overset{}{AQ}} = R^T = (\tilde{R}^T\ 0)$$

$$\Rightarrow AQ_2 = 0 \quad \Rightarrow \mathcal{R}(Q_2) \subset \mathcal{N}(A)$$

and $\dim Q_2 = n-m \Rightarrow \mathcal{R}(Q_2) = n-m$

Optimality conditions for EQP

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Let x_0 satisfy $Ax_0 = b$ (i.e. x_0 is a feasible point)

Let $Z \in \mathbb{R}^{n \times (n-m)}$, rank $(Z) = n-m$ s.t. $AZ = 0$

(basis for nullspace of A)

Then

x satisfies $Ax = b$ iff $x = x_0 + Zu$, for some $u \in \mathbb{R}^{n-m}$.

Plug this representation of feasible points into (EQP):

$$\min_{u \in \mathbb{R}^{n-m}} \frac{1}{2}(x_0 + Zu)^T Q (x_0 + Zu) + d^T (x_0 + Zu).$$

Unconstrained problem!

$$\Leftrightarrow \min_{u \in \mathbb{R}^{n-m}} \frac{1}{2} u^T Z^T Q Z u + (d + Qx_0)^T Z u + \text{cont.} \quad (*)$$

Theorem

i) u_* solves $(*)$ iff

- $Z^T Q Z u_* = - Z^T (d + Qx_0)$
- $Z^T Q Z$ is pos semi def.

ii) if $Z^T Q Z$ is pos def, then $(*)$ has a unique sol. given by:

$$u_* = -(Z^T Q Z)^{-1} Z^T (d + Qx_0)$$

Sol to EQP is then: $\boxed{x_* = x_0 + Z u_*}$

Why did we assume Q symm?

If Q is not symm, not symm part disappears when we look at quadratic form:

$$f(x) = x^T Q x = \frac{1}{2}\left(x^T Q x + x^T Q^T x\right)$$
$$= \frac{1}{2}\left(x^T \underbrace{(Q^T + Q)}_{\text{symm}} x\right)$$

$$\nabla f(x) = Qx + Q^T x$$

Application of constrained optimization to EQP

$$L(x, y) = \frac{1}{2} x^T Q x + d^T x + y^T (Ax - b)$$

First order nec. opt. cond → exercise: show this is equivalent to what we got with nullspace approach

$$\begin{cases} Qx + d + A^T y = 0 \\ Ax - b = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{pmatrix} Q & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -d \\ b \end{pmatrix}$$

Second order nec. opt. cond

Q pos semidef for all $v \in \mathcal{N}(A)$

$$\Leftrightarrow \quad Z^T Q Z \geq 0$$

Define the KKT matrix:

$$K = \begin{pmatrix} Q & A^T \\ A & 0 \end{pmatrix}$$

Theorem: K is invertible iff $Z^T Q Z$ is invertible

For small problems it is actually possible to form K and solve KKT system. However we still need to check that $Z^TQZ > 0$ to be sure to have a minimizer.

<u>How to solve KKT systems?</u>

- <u>Range Space Approach</u> (Gaussian elim.)

If Q is pos def

1st eq $\quad x = Q^{-1}(-A^Ty - d)$ (x variables are eliminated)

2n eq $\quad AQ^{-1}(-A^Ty - d) = b$

$$\Rightarrow \quad AQ^{-1}A^Ty = -b - AQ^{-1}d$$

works well if Q is easy to invert and there are few equality constraints then $AQ^{-1}A^T$ is easy to invert

(note: this matrix is dense in general, even if A, Q are sparse)

- <u>Null space Approach</u> Assumes representation Z of the nullspace and knowledge of some x_0 s.t. $Ax_0 = b$.

$$Z \in \mathbb{R}^{n \times (n-m)}, \ \operatorname{rank}(Z) = n - m$$

$$AZ = 0$$

solve $Z^TQZu = -Z^T(Qx_0 + d)$

set $x = x_0 + Zu$.

Use Cholesky for Z^TQZ : automatic check for 2nd order suff opt. cond.

Can also use iterative method (such as CG)

Inequality constrained problems

$$(QP) \begin{cases} \min \frac{1}{2} x^T Q x + x^T d \\ \text{s.t.} \quad a_i^T x = b_i \quad i \in E \\ \qquad a_i^T x \geq b_i \quad i \in I \end{cases}$$

Let x^* be a solution of (QP).

$$A(x^*) = \{ i \in E \cup I \mid a_i^T x_* = b_i \} = \text{active constraints}$$

If we knew $A(x^*)$ in advance we could solve

$$\min \frac{1}{2} x^T Q x + d^T x$$
$$\text{s.t.} \quad a_i^T x_i = b_i \,, i \in A(x_*)$$

if we were close enough to x_*, since strict inequalities remain strict ineq. in a neighborhood of x_*.

But $A(x^*)$ is unknown! Active set approaches try to guess it and improve guess with improved x^*.

Optimality conditions

Lagrangian:

$$L(x,y) = \frac{1}{2} x^T Q x + x^T d - \sum_{i \in E} y_i (a_i^T x - b_i) - \sum_{i \in I} y_i (a_i^T x - b_i)$$

Optimality conditions:

$$Q x + d - \sum_{i \in E} a_i y_i - \sum_{i \in I} a_i y_i = 0$$
$$y_i \geq 0 \,, i \in I$$
$$(b_i - a_i^T x) y_i = 0 \,, i \in I$$
$$a_i^T x = b_i \,, i \in E$$
$$a_i^T x \geq b_i \,, i \in I$$

Idea: solve a sequence of QP with equality constraints given by Working set

$$E \subset \mathcal{W}(x) \subset \mathcal{A}(x)$$

Take a step in which $\mathcal{W}(x)$ is the same

If boundary is hit, update $\mathcal{W}(x)$.

~ same flavor as simplex method, has exponential worst case complexity.

Active set method for QP:

0) Given feasible pt x_0. compute associated L.M. y_0

1) if x_0, y_0 satisfy optimality cdt° (KKT) STOP.

Select W_0 s.t.

$$E \subset W_0 \subset \mathcal{A}$$

set $k=0$

2) Set $(y_{k+1})_i = 0 \quad i \notin W_k$

Solve $\min \frac{1}{2} x^T Q x + x^T d$

s.t $a_i^T x = b_i \quad , i \in W_k$

$\Updownarrow$

Solve $\min \frac{1}{2} p^T Q p + (Qx_k + d)^T p$

s.t. $a_i^T p = 0, \quad i \in W_k$

$\Updownarrow$

Solve

$$\begin{bmatrix} Q & A_{W_k}^T \\ A_{W_k} & 0 \end{bmatrix} \begin{pmatrix} p \\ y_{W_k} \end{pmatrix} = \begin{bmatrix} -(Qx_k + d) \\ 0 \end{bmatrix}$$

3] a) If $p=0$ and $y_i \geq 0$, $i \in I$ STOP; x_k solves QP. (8)

b) If $p=0$ and $\exists q \in I$ s.t. $y_q < 0$ then:

$x_{k+1} = x_k$, $W_{k+1} = W_k \setminus \{q\}$

goto 4

c) If $p \neq 0$ and $x_k + p$ feasible for (QP)

$x_{k+1} = x_k + p$

$W_{k+1} = W_k$

goto 4

d) If $p \neq 0$ and $x_k + p$ NOT feasible for (QP):

compute largest α s.t.

$$a_i^T (x_k + \alpha p) \geq b_i, \quad i \in I \setminus W_k$$

$$\alpha = \min \left\{ \frac{b_i - a_i^T x_k}{a_i^T p} \;\middle|\; a_i^T p < 0; \; i \in I \setminus W_k \right\}$$

$x_{k+1} = x_k + \alpha p$

$W_{k+1} = W_k \cup \{r\}$

↑ index where min is attained

4] set $k = k+1$; goto 1].

Note

if $i \in I \setminus W_k$ & $a_i^T p \geq 0$ then: $\forall \alpha \geq 0$:

$a_i^T (x_k + \alpha p) \geq a_i^T x_k \geq b$

if $i \in I \setminus W_k$ & $a_i^T p < 0$ then:

$a_i^T (x_k + \alpha p) \geq b_i$

$\Leftrightarrow \quad \alpha \leq \dfrac{b_i - a_i^T x_k}{a_i^T p}$

Theorem:

i) Let Q be spd and let a_i, $i \in W_k$ be lin indep then EQP has a unique sol (p is unique)

ii) If a_i, $i \in W_k$ are lin indep then
a_i, $i \in W_{k+1}$ " " "

iii) If Q is spd then p computed in step 2 satisfies:

$$\underbrace{(Q x_k + d)}_{\text{grad of obj fun}}^T p < 0$$

$\rightsquigarrow$ p is a descent direction for unconstrained problem. But since we have constraints we may not go too far.