Things we ommitted:

i) You need to start with a b.f.p.

ii) How to avoid cycling in the case of degenerate vertices "Bland's rule"

iii) Only one column changes in A_B, there are ways to reuse previous solves (update LU decomp)

See Nocedal & Wright §13.4. (but not necessary to get it working)

Worst case scenario for Simplex method is to have to explore all vertices of polyhedron (feasible region), thus the complexity of the simplex method is exponential!

Fortunately in **most** applications one reaches solution roughly in linear time (in the number of rows)

Interior point methods are polynomial time algorithms to solve LP (and can be modified to solve other more general optimization problems) (~ 1980)

For LPs Simplex method is generally faster than IPM for small-medium sized problems.

Interior point method for LP

$$(LP)\quad \begin{cases} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$$

$x \in \mathbb{R}^n$
$A \in \mathbb{R}^{m \times n}$
$b \in \mathbb{R}^m$

Optimality conditions

$$(KKT)\quad \begin{cases} A^T y + z = c \\ A\ x = b \\ x_i z_i = 0 \quad i = 1..n \\ x, z \geq 0 \end{cases}$$

Denote $X = \operatorname{diag}(x_1, \dots, x_n)$
$Z = \operatorname{diag}(z_1, \dots, z_n)$
$\underline{e} = (1, \dots 1)^T$

Consider a slightly perturbed version of (KKT) with $\tau > 0$:

$$(1)\quad \begin{cases} A^T y + z = c \\ A x = b \\ x_i z_i = \tau, \ i = 1..n \iff Xz = \tau \underline{e} \\ x, z \geq 0 \end{cases}$$

The set of points (x_τ, y_τ, z_τ) is called the <u>central path</u>.

As $\tau \to 0$ if (x_τ, y_τ, z_τ) converges it must converge to a solution of (KKT).
If we follow the central path we are staying away from edges of polyhedron (since $\tau > 0 \Rightarrow x > 0$). This allows to take steps in potentially all directions.

<u>Theorem</u>: The following are equivalent:

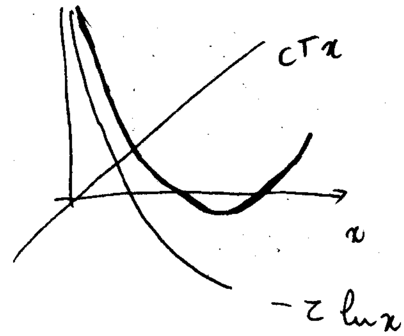
i) The problem (1) has a solution

ii) The primal barrier problem

$$\min \; c^T x - \tau \sum_{i=1}^{n} \ln(x_i)$$ ← penalizes being close to boundary

$$\text{s.t. } Ax = b$$
$$x > 0$$
↑ STRICT

has a solution



iii) The dual barrier problem

$$\max \; b^T y + \tau \sum_{i=1}^{n} \ln(z_i)$$

$$\text{s.t. } A^T y + z = c$$
$$z > 0$$

has a solution.

proof (ii ⇒ i, other directions can be proven similarly)

Define :

$$L(x, y, z) = c^T x - \tau \sum_{i=1}^{n} \ln(x_i) - (Ax - b)^T y - x^T z$$

$$\nabla L(x, y, z) = c - \tau X^{-1} e - A^T y -$$

If x solves ii, then $\exists\, y, \hat{z}$ s.t.

$$c - \tau X^{-1} e = A^T y + \hat{z}$$
$$\hat{z} \geq 0$$
$$x_i \hat{z}_i = 0 \iff X\hat{z} = 0$$
$$Ax = b, \; x > 0$$

Letting $z = \tau X^{-1} e$:

$$\Rightarrow X A^T y + \underbrace{X\hat{z}}_{=0} = Xc - \underbrace{\tau e}_{Xz} \Rightarrow X(A^T y + z) = Xc$$

$$\iff A^T y + z = c.$$

Now we need to figure out when does primal barrier problem has a solution.

$$\mathcal{F} = \{(x,y,z) \mid Ax=b,\ A^Ty+z=c,\ x,z \geq 0\}$$

$$\mathcal{F}^\circ = \{(x,y,z) \mid Ax=b,\ A^Ty+z=c,\ x,z > 0\}$$

strict

Theorem: If $\mathcal{F}^\circ \neq \phi$ then the primal barrier problem has a solution for all $\tau > 0$.

Note: here is an example where $\mathcal{F}^\circ = \phi$, in which (1) does not have a sol:

$$\begin{aligned} \min\ & x_1 + x_2 \\ \text{s.t.}\ & x_1 + x_2 = 0 \\ & x \geq 0 \end{aligned}$$

Sol is $(0,0)$. We cannot have $x > 0$ feasible because constraints imply $x=0$.

How to solve (1) for given $\tau > 0$:

$w = (x,y,z)$

Let $F_\tau(w) = \begin{bmatrix} A^Ty + z - c \\ Ax - b \\ XZe - \tau e \end{bmatrix}$

Quadratic map of w

Solving $F_\tau(w) = 0 \sim$ solving (1)

Use Newton's method for NL eq:

$$F_\tau'(w) = \begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix}$$

(Jacobian)

Lemma: If $x>0, z>0$ and if $A \in \mathbb{R}^{m \times n}$ has full rank m, then $F_\tau'(w)$ is invertible.

Thus we can use Newton's method to compute update $\Delta w = (\Delta x, \Delta y, \Delta z)$ solving:

$$F_\tau'(w)\Delta w = -F(w)$$

Assume current iterate w is s.t. $A^T y + z = c$ and $Ax = b$.

then $w + t\Delta w = (x + t\Delta x, y + t\Delta y, z + t\Delta z)$ satisfies:

$$A^T(y + t\Delta y) + z + t\Delta z = c \quad \text{and} \quad A(x + t\Delta x) = b.$$

proof: $F_\tau'(w)\Delta w = -F_\tau(w) \Rightarrow \begin{array}{l} A^T\Delta y + \Delta z = 0 \\ A\Delta x = 0 \end{array}$

$= \begin{pmatrix} 0 \\ 0 \\ ? \end{pmatrix}$ ← feasibility

Thus if we start with a feasible point we remain feasible

Centrality parameter

$\tau = \sigma\mu$, where $\sigma \in (0,1)$ = centering parameter

and $\mu = \dfrac{x^T z}{n}$ = duality gap (indicates progress to sol ~ how far we are down central path)

Feasible interior point algo:

Let $w_0 \in \mathcal{F}^\circ = \{x, y, z \mid A^T y + z = c,\ Ax = b,\ x, z > 0\}$

Let $\varepsilon \in (0,1)$, $k=0$

For $k = 0, 1, \ldots$

if $\mu_k = \dfrac{x_k^T z_k}{n} < \varepsilon$ STOP

Choose $\sigma_k \in (0,1)$, let $\tau_k = \sigma_k \mu_k$

(*) solve $F_{\tau_k}'(w_k)\Delta w = -F_{\tau_k}(w_k)$

Set $w_{k+1} = w_k + t_k \Delta w$ s.t. where t_k chosen s.t. $x_{k+1} > 0$ & $z_{k+1} > 0$

maintains positivity to remain in $\mathcal{F}_0$.

There are also infeasible IPM, same thing but $w_0 \notin \mathcal{F}_0$, but we do require $x_0 > 0, z_0 > 0$. Analysis is harder though.

Lemma: Let Δw be sol to $(*)$.

Define

$$\mu_k(t) = \frac{(x_k + t\Delta x)^T (z_k + t\Delta z)}{n}$$

then $\Delta x^T \Delta z = 0$ Duality gap $= \frac{x_k^T z_k}{n}$

and $\mu_k(t) = (1 - t(1 - \sigma_k))\mu_k$

σ_k = centrality parameter

Proof: $A^T \Delta y + \Delta z = 0 \Rightarrow \boxed{\Delta x^T \Delta z = -\underbrace{\Delta x^T A^T}_{=0} \Delta y = 0}$

If $\sigma_k = 0$: step is made towards solution of optimality system (KKT)
⇝ we might end trapped closed to the boundary, forced to take small steps (t_k small)
Worst than simplex!

If $\sigma_k = 1$ then $\mu_{k+1} = \mu_k$ = constant. We're just going to central path, but we don't really want central path.