Things are commited:

i) You need to start with a bfs.

ii) How to avoid cycling in the case of degenerate vertex. "Bland's rule"

iii) Only one column change in $A^T$, there are ways to reuse previous solves (update LU decomp)

See Nocedal & Wright §13.4. (but not necessary to get it working)

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Worst case scenario for Simplex method is to have to explore all vertices of polyhedron (feasible region), thus the complexity of the simplex method is exponential!

Fortunately in most applications one reaches solution roughly in linear time (on the number of rows)

Interior point methods are polynomial time algorithms to solve LPs (and can be modified to solve other more general optimization problems) (~ 1980)

For LPs Simplex method is generally faster than IPM for small-medium sized problems.
Introduce point method for LP

\[
\begin{align*}
\text{(LP)} \quad & \min \quad c^T x \\
\text{st.} \quad & A x = b \\
& x \geq 0
\end{align*}
\]

\[A \in \mathbb{R}_{mn}^n, \quad b \in \mathbb{R}^m\]

Optimality Conditions

\[
\begin{align*}
& A^T y + z = c \\
& A x = b \\
& x, y, z \geq 0 \quad i = 1 \ldots n
\end{align*}
\]

Denote \(X = \text{diag}(x_1, \ldots, x_n)\)
\(Z = \text{diag}(z_1, \ldots, z_n)\)
\(e = (1, \ldots, 1)\)^T

Consider a slightly perturbed version of (KKT) with \(\tau > 0\)

\[
\begin{align*}
& A^T y + z = c \\
& A x = b \\
& x_i z_i = \tau \quad i = 1 \ldots n \quad (\tau) \\
& x, z \geq 0
\end{align*}
\]

The set of points \((x_\tau, y_\tau, z_\tau)\) is called the central path.

As \(\tau \to 0\) if \((x_\tau, y_\tau, z_\tau)\) converges it must converge to a solution of (KKT).

If we follow the central path we are staying away from edges of polyhedron (since \(\tau > 0 \Rightarrow x > 0\)). This allows to take steps in potentially all directions.

**Theorem:** The following are equivalent:

1. The problem (1) has a solution
ii) The primal barrier problem

\[ \min c^T x - 2 \sum_{i=1}^{n} \ln(x_i) \]

s.t. \[ A x = b \]
\[ x > 0 \]

has a solution.

iii) The dual barrier problem

\[ \max b^T y + 2 \sum_{i=1}^{n} \ln(z_i) \]

s.t. \[ A^T y + z = c \]
\[ z > 0 \]

has a solution.

proof \( \Rightarrow i \), other directions can be proven similarly.

Define:

\[ L(x, y, z) = c^T x - 2 \sum_{i=1}^{n} \ln(x_i) - (Ax - b)^T y - x^T z \]

\[ \nabla L(x, y, z) = c - 2 x^T e - A^T y \]

If \( x \) solves ii, then \( \exists y, \hat{z} \) s.t.

\[ c - 2 x^T e = A^T y + \hat{z} \]
\[ \hat{z} \geq 0 \]
\[ x_i \hat{z}_i = 0 \Rightarrow x^T \hat{z} = 0 \]
\[ A x = b, \ x > 0 \]

Letting \( \hat{z} = 2 x^T e \):

\[ x A^T y + x^T \hat{z} = x c - 2 x e \Rightarrow x (A^T y + \hat{z}) = x c \]
\[ x \hat{z} = 0 \Rightarrow x^T \hat{z} = 0 \]

\( \Rightarrow A^T y + \hat{z} = c \).
Now we need to figure out when does primal barrier problem have a solution.

\[ F = \{ (x, y, z) \mid Ax = b, A^Ty + y = c, x, z \geq 0 \} \]

\[ F^0 = \{ (x, y, z) \mid Ax = b, A^Ty + y = c, x, z > 0 \} \]

**Theorem.** If \( F^0 \neq \emptyset \) then the primal barrier problem has a solution for all \( z > 0 \).

Note: here is an example where \( F^0 = \emptyset \), in which (1) does not have a solution:

\[
\begin{align*}
\text{min} & \quad x_1 + x_2 \\
\text{st.} & \quad x_1 + x_2 = 0 \\
& \quad x \geq 0
\end{align*}
\]

Solve \((0,0)\). We cannot have \( x > 0 \) feasible because constraints imply \( x = 0 \).

How to solve (1) for given \( z > 0 \):

\[ \omega = (x, y, z) \]

Let \( F_e(\omega) = \begin{bmatrix} A^Ty + y - c \\ Ax - b \\ xze - ze \end{bmatrix} \)

Use Newton's method for NL eq:

\[ F_e'(\omega) = \begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \]

(Quadratic map of \( fe \))

Solving \( F_e(\omega) = 0 \) \& solving (1)
Lemma: If $x > 0$ and if $A \in \mathbb{R}^{m \times n}$ has full rank $m$, then $F(x)$ is invertible.

Thus we can use Newton's method to compute update $\Delta w = (\Delta x, \Delta y, \Delta z)$ solving:

$$F'(w) \Delta w = -F(w)$$

Assume current iterate $w$ is in $\mathcal{F}$, $A^T y + z = 0$ and $Ax = b$.

Then $w + \delta w = (x + \delta x, y + \delta y, z + \delta z)$ satisfies:

$$A^T(y + \delta y) + z + \delta z = 0 \quad \text{and} \quad A(x + \delta x) = b.$$  

Proof: $F'(w) \Delta w = -F(w) \Rightarrow A^T \Delta y + \Delta z = 0$

Thus if we start with a feasible point we remain feasible.

Centrality parameter

$$\tau = \frac{\sigma m}{\mu}, \quad \text{where} \quad \sigma \in (0, 1) = \text{centering parameter}$$

and $\mu = \frac{x^T z}{m} = \text{duality gap}$ (indicates progress toward how far we are down central path)

Feasible interior point algo:

Let $w_0 \in \mathcal{F} = \{x, y, z \mid A^T y + z = 0, Ax = b, x, y > 0\}$

Set $\varepsilon \in (0, 1), k = 0$

For $k = 0, 1, \ldots$

$$\frac{\mu_k}{\tau_k} \leq \varepsilon \quad \text{STOP}$$

Choose $\tau_k \in (0, 1)$, let $\tau_k = \frac{\mu_k}{\tau_k}$

Solve $F_{\tau_k}(w_k) \Delta w = -F_{\tau_k}(w_k)$$$

Set $w_{k+1} = w_k + \delta \Delta w$ s.t. where $\delta$ chosen so $x_{k+1} > 0, z_{k+1} > 0$ and $\Delta w > 0$. 

maintains positivity to remain in $\mathcal{F}_0$. 


There are also infeasible IPM, same thing but \( \nu \neq 0 \), but
we do require \( x_0 > 0, y_0 > 0 \). Analysis is harder though.

**Lemma:** Let \( \Delta w \) be sol to \((\ast)\).

Define

\[
\mu_k(t) = \frac{((x_k + t\Delta x)^T (y_k + t\Delta y))}{n}
\]

Then \( \Delta x^T \Delta y = 0 \)

Duality gap = \( \frac{x_k^T y_k}{n} \)

and \( \mu_k(t) = (1 - t(1-\sigma_t)) \mu_k \)

\( \sigma_t \) = centrality parameter

**Proof:**

\[
A^T \Delta y + \Delta z = 0 \implies \Delta x^T \Delta z = -\Delta x^T A^T A \Delta y = 0
\]

If \( \sigma_t = 0 \): step is made towards solution of optimality system \((\kkt)\)

\( \implies \) we might end trapped close to the boundary, forced to
take small steps (\( t_k \) small)

Worse than simplex!

If \( \sigma_t = 1 \) then \( \mu_{k+1} = \mu_k = \text{constant} \). We're just going to central
path, but we don't really want central path.