Recall:

primal LP
min $c^T x$
s.t. $Ax = b$
$x \geq 0$

dual LP
max $b^T y$
s.t. $A^T y \leq c$

optimality cdts:

$$\begin{aligned} Ax &= b \\ A^T y + z &= c \\ x \geq 0,\ z &\geq 0 \\ x^T z &= 0 \end{aligned}$$

$\mathcal{F} = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$
= polyhedron in std form

<u>x is a vertex</u> if:

$x = \lambda x_1 + (1-\lambda) x_2$, $\lambda \in (0,1)$, $x_1, x_2 \in \mathcal{F}$
$\Rightarrow x_1 = x_2 = x$

<u>Theorem</u> Let $\mathcal{F}$ be a polyhedron in std form then $x \in \mathcal{F}$ is a vertex iff columns A_i of A for which $x_i > 0$ are lin indep.

proof (here only $\Leftarrow$ part)

Let $x \in \mathcal{F}$, $I = \{i \mid x_i > 0\}$, $A_i, i \in I$ lin indep

Let $x^1, x^2 \in \mathcal{F}$ and $\lambda \in (0,1)$ s.t.

$$x = \lambda x^1 + (1-\lambda) x^2$$

Since $x^1, x^2 \geq 0$ and $x_i = 0$ for $i \notin I$ here:

$$x_i^2 = x_i^1 = 0 \quad \text{for } i \notin I$$

Thus:

$$0 = b - b = Ax^1 - Ax^2 = \sum_{i \in I} \overbrace{A_i}^{\text{lin indep}} (x_i^1 - x_i^2) \Rightarrow x_i^1 = x_i^2,\ i \in I$$

$$\Rightarrow x^1 = x^2.$$

Def Let $\mathcal{F}$ be a polyhedron in std form. We say $x \in \mathcal{F}$ is a basic feasible point if there is some index set $\mathcal{B} \subset \{1..n\}$ with $|\mathcal{B}| = m$ and $x_j = 0$ for $j \notin \mathcal{B}$ and $A_i, i \in \mathcal{B}$ are lin indep

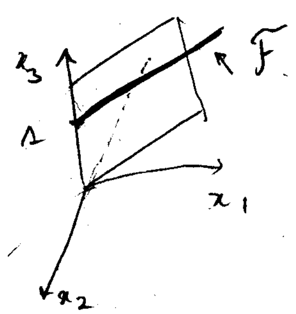
Theorem: Let $\mathcal{F}$ be a polyhedron in std form s.t.

$$A \in \mathbb{R}^{m \times n} \; ; \; \text{rank}(A) = m.$$

x is a vertex of $\mathcal{F} \iff x$ is a b.f.p.

Note: representation of a vertex as a b.f.p. is not unique. We can have degenerate b.f.p.:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_1, x_2, x_3 \geq 0.$$



vertex $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a b.f.p w/ $\mathcal{B} = \{1, 3\}$ or $\mathcal{B} = \{2, 3\}$

Theorem: $\mathcal{F}$ is a polyhedron in std form, $A \in \mathbb{R}^{m \times n}$, rank$(A) = m$ then:

- $\mathcal{F} \neq \emptyset \Rightarrow \mathcal{F}$ contains a b.f.p.
- $\mathcal{F}$ has finitely many b.f.p.
- If $\min c^T x$ s.t. $Ax = b$, $x \geq 0$ has a solution, then one of the sol is b.f.p.

Simplex Method:

let x be a b.f.p. and let $\mathcal{B} \subset \{1..n\}$ be the corresp. index set.

let $I = \mathcal{B}^c = \{1..n\} \setminus \mathcal{B}$

Since $x_I = 0$

$$Ax = A_B x_B = b \Rightarrow x_B = A_B^{-1} b$$

How do we check x is a minimum? Use optimality conditions:

$$\begin{pmatrix} A_B^T \\ A_I^T \end{pmatrix} y + \begin{pmatrix} z_B \\ z_I \end{pmatrix} = \begin{pmatrix} c_B \\ c_I \end{pmatrix} \qquad (\nabla L = 0)$$

$$x^T z = x_B^T z_B = 0 \qquad \text{(Compl)}$$

Setting $z_B = 0$ to satisfy compl. cdts we get.

$$\begin{cases} y = A_B^{-T} c_B \\ z_I = c_I - A_I^T y \end{cases}$$

If $z_I \geq 0$, then x_I is optimal (since all optimality cond. hold)

What to do if current point is not a min?

then $\exists\, q \in I$ s.t. $z_q < 0$

Let s_B be s.t.

$$A_B s_B = -A_q$$

and s_I be s.t.

$$s_i = \begin{cases} 1 & \text{if } i = q \\ 0 & \text{if } i \in I \setminus \{q\} \end{cases}$$

then:

$$A \underbrace{\begin{pmatrix} s_B \\ s_I \end{pmatrix}}_{s = \text{step}} = A_B s_B + A_q = 0$$

Now let's see what happens with function values

$$c^T(x+ts) = c_B^T x_B + t\, c_B^T s_B + t c_q$$
$$= c_B^T x_B - t\, c_B^T A_B^{-1} A_q + t c_q$$

Recall $z_I = c_I - A_I^T A_B^{-T} c_B$

$z_q = c_q - A_q^T A_B^{-T} c_B < 0$ (assumed)

$$\Rightarrow c^T(x+ts) = c_B^T x_B + \underbrace{t z_q}_{<0} \leq c_B^T x_B = c^T x$$

So this step makes function values decrease.

We need to ensure $x+ts \geq 0$, take largest t for which

$$x_B + t s_B \geq 0$$

- if $s_B \geq 0$, then LP is unbounded and does not have a sol since: $x_B + t s_B \geq 0 \ \forall t$, and thus $t z_q \to -\infty$.
- Otherwise there must be some $s_i < 0$, $i \in \mathcal{B}$:

$$t = \min_{\substack{i \in \mathcal{B} \\ s_i < 0}} \frac{-x_i}{s_i} \qquad (*)$$

New iterate:

$$x_{new} = x + ts$$

Let $r \in \mathcal{B}$ be index for which min in $(*)$ is taken:

$$\frac{-x_r}{s_r} = \min_{\substack{i \in \mathcal{B} \\ s_i < 0}} \frac{-x_i}{s_i}$$

$$\mathcal{B}_{new} = \mathcal{B} \cup \{q\} \setminus \{r\}$$

was non zero becomes zero

was zero, becomes non zero

Claim: x_{new} is a b.f.p. with corresp. idx set $\mathcal{B}_{new}$.

first: $(x_{new})_i = 0 \quad i \notin \mathcal{B}_{new}$ (by construction)

$$0 = \sum_{i \in \mathcal{B}_{new}} A_i \gamma_i$$

$$= \sum_{\substack{i \in \mathcal{B} \\ i \neq r}} A_i \gamma_i + \gamma_q \underbrace{A_q}_{= -A_B s_B}$$

$$= \sum_{\substack{i \in \mathcal{B} \\ i \neq r}} A_i (\gamma_i - s_i \gamma_q) - \gamma_q s_r A_r$$

Using that A_i, $i \in \mathcal{B}$ are lin indep: $\begin{cases} \gamma_i - s_i \gamma_q = 0, & i \in \mathcal{B}, i \neq r \\ \gamma_q s_r = 0 \end{cases}$

$$\Rightarrow \begin{cases} \gamma_i = 0, & i \in \mathcal{B}, i \neq r \\ \gamma_q = 0 & \text{(since index } r \text{ is s.t. } s_r < 0) \end{cases}$$

$\Rightarrow \gamma_i = 0, \quad i \in \mathcal{B}_{new}$. QED.

SIMPLEX ALGO: Given bfp x, $\mathcal{B} \subset \{1..n\}$

1. $y = A_B^{-T} c_B$

   $z_I = c_I - A_I^T \underbrace{A_B^{-T} c_B}_{y}$

2. if $z_I \geq 0$, stop. optimal sol. x found.
3. select $q \in I$ s.t. $z_q < 0$
4. Compute descent dir: $s_B = -A_B^{-1} A_q$

   $s_I: s_i = \begin{cases} 1 & \text{if } i = q \\ 0 & \text{if } i \in I \setminus \{q\} \end{cases}$

5. If $s_i \geq 0$, $i \in \mathcal{B}$. STOP LP has unbdd obj fun (no sol)
6. Compute $r \in \mathcal{B}$ s.t. $-\frac{x_r}{s_r} = \min_{\substack{i \in \mathcal{B} \\ s_i < 0}} -\frac{x_i}{s_i}$
7. $x \leftarrow x - \frac{x_r}{s_r} s$; $\mathcal{B} \leftarrow (\mathcal{B} \cup \{q\}) \setminus \{r\}$