Recall (NLP) $\begin{cases} \min f(x) \\ \text{s.t. } c_i(x) = 0, \ i = 1..m_E \\ \quad\ c_i(x) \geq 0, \ i = m_E+1..m \end{cases}$ $\quad f: \mathbb{R}^n \to \mathbb{R}$, $c_i: \mathbb{R}^n \to \mathbb{R}$

The first order optimality conditions are:

$\bar{x}$ local min of (NLP) $\Rightarrow \exists \bar{\lambda} \in \mathbb{R}^m$ s.t. $\begin{cases} c_i(\bar{x}) = 0, & i = 1..m_E \\ \bar{\lambda}_i c_i(\bar{x}) = 0, & i = m_E+1, \dots, m \\ \nabla_x L(\bar{x}, \bar{\lambda}) = 0 \\ \bar{\lambda}_i \leq 0, & i = m_E+1, \dots, m \end{cases}$

$\left(L(x,\lambda) = f(x) + Dc(x)^T \lambda = f(x) + \sum_{i=1}^{m} \lambda_i \nabla c_i(x) \right)$

(nonlinear syst of eq: m+n eq^{ns} for m+n unknowns)

$$G(x,\lambda) = \begin{bmatrix} c_E(x) \\ \Lambda_I c_I(x) \\ \nabla_x L(x,\lambda) \end{bmatrix} \qquad \begin{aligned} E &= \{1 \dots m_E\} \\ I &= \{m_E+1 \dots m\} \\ \Lambda_I &= \operatorname{diag}(\lambda_{m_E+1}, \dots, \lambda_m) \end{aligned}$$

Jacobian:

$$DG(x,\lambda) = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} \underset{\downarrow}{x} & \underset{\downarrow}{\lambda_E} & \underset{\downarrow}{\lambda_I} \end{array}$$

$$DG(x,\lambda) = \begin{bmatrix} Dc_E(x) & 0 & 0 \\ \Lambda_I Dc_I(x) & 0 & \operatorname{diag}(c_I(x)) \\ \nabla_{xx} L(x,\lambda) & Dc_E^T(x) & Dc_I^T(x) \end{bmatrix}$$

<u>When is DG invertible?</u> (for example to use Newton's method for NLeq)

$I_+ = \{ i \in I \mid c_i(x) > 0 \}$ strict ineq. inactive constraints

$I_0 = I \setminus I_+$: active constraints

Partition DG using I_0 & I_+: $\quad \underset{\downarrow}{x} \quad \underset{\downarrow}{\lambda_E} \quad \lambda_{I_0} \quad \lambda_{I_+}$

$$DG(x,\lambda) = \left[\begin{array}{c|c|c|c} Dc_E(x) & 0 & 0 & 0 \\ \hline \Lambda_{I_0} Dc_{I_0}(x) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \operatorname{diag}(c_{I_+}(x)) \\ \hline \nabla_{xx} L(x,\lambda) & Dc_E^T(x) & Dc_{I_0}^T(x) & Dc_{I_+}^T(x) \end{array}\right]$$

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- If DG is invertible then necessarily: $\boxed{\lambda_i < 0 \quad i \in I_0}$

(if there is one $\lambda_i = 0$, $i \in I_0$, then DG has a row of zeros $\Rightarrow$ singular)

This is called <u>strict complementarity</u>:

only one of $c_i(x)$ and λ_i can be zero.

DG invertible $\Leftrightarrow$ the following matrix is invertible:

$$\begin{pmatrix} Dc_E(x) & 0 & \\ \Lambda_{I_0} Dc_{I_0}(x) & 0 & \\ \nabla_{xx} L(x,\lambda) & Dc_E^T(x) & Dc_{I_0}^T(x) \end{pmatrix}$$

Thus: DG invertible $\Rightarrow$ $Dc_E(x)$ & $\Lambda_{I_0} Dc_{I_0}(x)$ are full row rank

$\Leftrightarrow$ $Dc_E(x)$ & $Dc_{I_0}(x)$

(if we have strict compl.)

$\Leftrightarrow$ ∇c_i, $i \in E \cup I_0 = \mathcal{A}(x)$ are lin indep

$\Leftrightarrow$ LICQ for x.

Also let $d \in \mathbb{R}^n$ s.t. $\nabla c_i(x)^T d = 0$ for $i \in E \cup I_0 = \mathcal{A}(x)$

i.e. $d \in$ "nullspace of linearized active constraints"

then:

$$(0, 0, d^T)\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix} = (d^T \nabla_{xx} L(x,\lambda), 0, 0)\begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}$$

$$= d^T \nabla_{xx} L(x,\lambda) d \geq 0$$

by 2nd order nec. cdto.

Thus if DG is invertible we must have $d^T \nabla_{xx} L(x,\lambda) d > 0$

Can prove the other way around:

$$\left.\begin{array}{l} \nabla c_i(x),\ i \in \mathcal{A}(x) \text{ lin indep} \\ \text{strict complementarity} \\ \text{2nd order suff} \end{array}\right\} \Rightarrow DG \text{ is invertible}$$