Hyperbolic Equations

Examples:

- Wave equation: $\frac{1}{c^2} u_{tt} = \Delta u$ appears in acoustics, electromagnetics, seismics, ...

- Advection equation: $u_t + \underline{v} \cdot \nabla u = 0$ models transport of a material at a steady velocity $\underline{v}$.

Here we shall focus on 1D advection equation:

$$\begin{cases} u_t + a u_x = 0 \\ u(x,0) = \eta(x) \end{cases}, \quad \text{where } a = \text{constant}$$

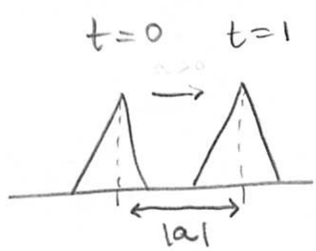
Exact solution: $u(x,t) = \eta(x - at)$

check: $u(x,0) = \eta(x)$

$u_x(x,t) = \eta'(x-at)$

$u_t(x,t) = -a\,\eta'(x-at)$

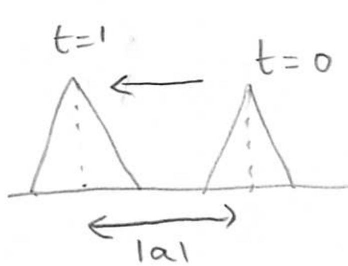
$a > 0$:



Solution is a "pulse" traveling at speed a. Sign of a gives direction of propagation.

$|a|$ = distance traveled in one time unit.

$a < 0$



First approach that comes to mind is to use the following approximations:

$$u_x(x,t) = \frac{u(x+h,t) - u(x-h,t)}{2h} + O(h^2)$$

= centered differences in space

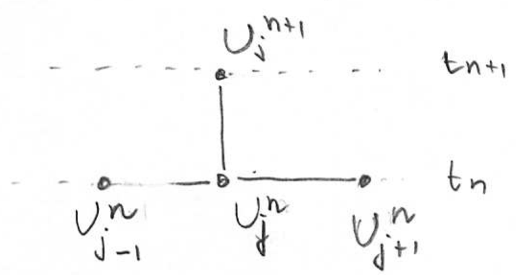
$$u_t(x,t) = \frac{u(x,t+k) - u(x,t)}{k} + O(k)$$

= forward differences in time.

$\leadsto$ numerical scheme: (<u>notation</u>: $U_j^n \approx u(x_j, t_n)$)

$$\boxed{\frac{U_j^{n+1} - U_j^n}{k} = -\frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)} \quad (*)$$

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$



This numerical scheme is not practical because of stability reasons. (as we shall see later)

<u>Lax Friedrichs method</u>

$$\boxed{U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)}$$

(like (*) but with U_j^n replaced by spatial average $\dfrac{U_{j+1}^n + U_{j-1}^n}{2}$)

(L-F. method is not used much in practice because of accuracy)

L.F. is Lax-Richtmyer stable and convergent provided:

$$\boxed{\left|\frac{ak}{h}\right| \leq 1} \quad \text{(CFL)}$$

This is a rather large time step compared to what we had for parabolic problems (where $k = O(h^2)$).

CFL = Courant, Friedrichs, Lewy condition

$\Leftrightarrow$ distance traveled in a time step $\quad$ cell size

$$|ak| \quad < \quad h$$

Another way of looking at this condition:

$$u_t = -a\, u_x$$

$\Leftrightarrow$ solution changes in time (t) <u>a times</u> faster than in space (x)

$\Leftrightarrow$ temporal resolution (k) must be at least <u>a times</u> smaller than spatial resolution (h)

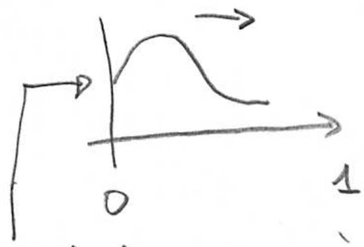
To shed some light on this and other methods for hyperbolic equations, we look at:

Method of Lines

Consider the advection equation together w/: the following B.C. :

$$\begin{cases} u_t + a u_x = 0 \\ u(0,t) = g_0(t) \end{cases}$$

if $a>0$: This is an "inflow" boundary condition



think of $g_0(t)$ as a 'stylus' writing on paper moving at velocity a in the indicated direction.

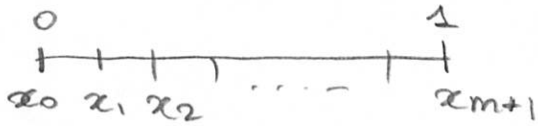
We use this B.C. rather than the initial condition (Cauchy problem) we started with because it is more adapted to bounded domains.

Note: if $a<0$ then inflow B.C. needs to be at $x=1$ because the direction of propagation is reversed.

To study stability, we make a further simplification and assume we have periodic boundary conditions:

$$u(0,t) = u(1,t) \quad \Leftrightarrow (\text{inflow at } x=0) = (\text{outflow at } x=1)$$

By the way the solution we obtain happens to be the same if we had assumed a Cauchy data (I.C. $u(x,0)=\eta(x)$) periodic (with period 1).



Let $U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_{m+1}(t) \end{bmatrix}$ where $U_j(t) \approx u(x_j, t)$ and we have kept only values of u at nodes that are not determined by B.C.

Semi discretizing advection eq in space with centered finite differences we obtain the (linear) system of ODEs:

$$\begin{cases} U_j'(t) = -\dfrac{a}{2h}\left(U_{j+1}(t) - U_{j-1}(t)\right), & \text{for } j = 2, \dots, m \\ U_1'(t) = -\dfrac{a}{2h}\left(U_2(t) - U_{m+1}(t)\right), & j = 1 \\ U_{m+1}'(t) = -\dfrac{a}{2h}\left(U_1(t) - U_m(t)\right), & j = m+1 \end{cases}$$

or in system form:

$$U'(t) = A\,U(t)$$

where

$$\boxed{A = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & \ddots & \ddots & 1 \\ 1 & & & -1 & 0 \end{bmatrix} \in \mathbb{R}^{m+1 \times m+1}}$$

A = <u>skew symmetric matrix</u> : $A^T = -A$

$\Rightarrow$ all its eigenvalues are <u>purely imaginary</u>.

[<u>proof</u>: is very similar to showing all eigenvalues of a symmetric matrix $A = A^T$ are real

let (λ, x) = eigenpair of A with $x^* x = 1$.

Here $x^* \equiv \overline{(x^T)}$.

$Ax = \lambda x \Rightarrow \boxed{x^* A x = \lambda x^* x = \lambda}$

$\Downarrow^*$

$(Ax)^* = (\lambda x)^* \rightarrow x^* A^* = \bar{\lambda} x^* \Rightarrow x^* A^* x = \bar{\lambda} x^* x$

$\Rightarrow \boxed{-x^* A x = \bar{\lambda}}$

therefore $\bar{\lambda} = -\lambda$ i.e. λ is imaginary.]

A = <u>circulant matrix</u> meaning a row can be obtained from previous one by a circular shift.

$\Rightarrow$ eigenvectors of A (and of any circulant matrix) are

$$u_j^p = e^{2i\pi p j h}, \quad p = 1, 2, \dots m+1 \qquad h = 1, 2, \dots, m+1$$

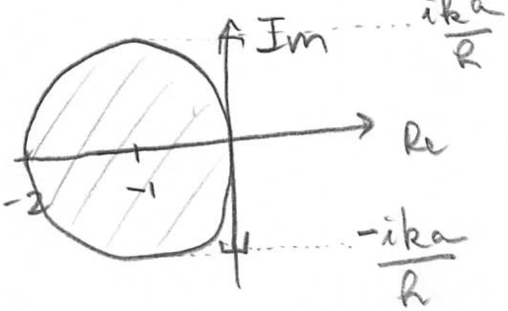
and $\lambda_p(A) = -\dfrac{ia}{h} \sin(2\pi p h)$ (no need to remember!)

i.e. eigenvalues of A lie on interval $\left[-\dfrac{ia}{h}, \dfrac{ia}{h}\right]$ on imaginary axis

Forward Euler: (first method we saw)

$$\boxed{\frac{U_j^{n+1} - U_j^n}{k} = -\frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)}$$

Absolute Stability region for Euler: $|1+k\lambda| < 1$



Eigenvalues of A always fall outside abs. stab region.

$\Rightarrow$ method is <u>not stable</u> because eigenvalues $k\lambda$ do not lie in absolute stability region when we keep $\frac{k}{h}$ fixed.

<u>Note</u>: if we let $k \to 0$ faster than h then $k\lambda_p \to 0$ so we get <u>convergent</u> method.

We can show this using previous theorem:

(Lax-Richtmyer stable & consistent) $\Leftrightarrow$ convergent

Say we take $k = h^2$

Look at:

$$B = I + kA$$

$$\Big|\underbrace{1}_{\in \mathrm{Re}} + \underbrace{k\lambda_p}_{\mathrm{Im}}\Big|^2 = 1 + |k\lambda_p|^2 \leq 1 + \left(\frac{ka}{h}\right)^2$$

$$\underset{k=h^2}{\leq} 1 + a^2h^2 = 1 + a^2k$$

Thus we obtain bound:

$$\|I + kA\|_2^2 \le 1 + a^2 k$$

$$\Rightarrow \|(I+kA)^n\|_2 \le (1+a^2k)^{\frac{n}{2}} \overset{(*)}{\le} e^{a^2 kn/2} \underset{\substack{\uparrow \\ nk \le T}}{\le} e^{a^2T/2}$$

$\Rightarrow \|B^n\|$ is uniformly bounded for $nk \le T$.

Inequality $(*)$ comes from:

$$(1+t)^\alpha \le e^{\alpha t} \qquad \text{when } \alpha > 0,\ t > 0$$

which can be shown using calculus or with Taylor:

$$(1+t)^\alpha = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2} t^2 + \dots$$

$$e^{\alpha t} = 1 + \alpha t + \frac{(\alpha t)^2}{2!} + \dots$$

Here is another numerical method:

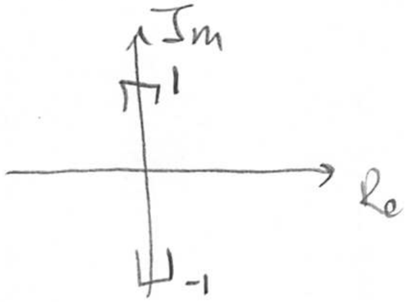
<u>Leapfrog</u>: use centered differences in time too:

$$\boxed{\frac{U_j^{n+1} - U_j^{n-1}}{2k} = -\frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)}$$

≡ 3 step, explicit method

≡ Second order in both space and time

Absolute stability region for this method (also called midpoint rule) is segment $[-1,1]$ on imaginary axis



$$\Rightarrow \quad -1 \leq \mathrm{Im}\, k\lambda_p \leq 1$$

$$\Rightarrow \quad \left|\frac{ak}{h}\right| \leq 1 \quad \text{(i.e. CFL cond.)}$$

However:

- $k\lambda_p$ is always right on boundary of absolute stability region.
  → <u>marginal stability</u> meaning small perturbations could lead to us falling outside abs. stab. region.
- all modes (frequencies) in sol are preserved (no decay or growth) <u>non-dissipative</u> method
- not all modes travel at same velocity: <u>dispersive method</u>

<u>Lax-Friedrichs</u>: revisited.

$$U_j^{n+1} = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

We rewrite the spatial average term as follows.

$$\frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) = U_j^n + \frac{1}{2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right)$$

$$\Rightarrow \quad U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \underbrace{\frac{1}{2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right)}_{\sim \text{discrete Laplacian}}$$

Or to reveal more structure we rewrite L.F. as:

$$\frac{U_j^{n+1} - U_j^n}{k} + a\,\frac{U_{j+1}^n - U_{j-1}^n}{2h} = \frac{h^2}{2k}\,\frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}$$

$\Rightarrow$ looks like solution to <u>advection-diffusion</u> eq:

$$u_t + a u_x = \underbrace{\varepsilon u_{xx}}_{\substack{\text{method incorporates some dissipation} \\ \text{for stability purposes.}}}, \quad \text{w/ } \varepsilon = \frac{h^2}{2k}$$

L.F. can be obtained by doing MOL on adv.-diff. eq and then using Euler's method:

MOL on adv. diff eq:

$$U'(t) = A_\varepsilon U(t)$$

where

$$\mathbb{R}^{m+1 \times m+1} \ni A_\varepsilon = \underbrace{-\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & \ddots & \ddots & 1 \\ 1 & & & -1 & 0 \end{bmatrix}}_{A} + \frac{\varepsilon}{h^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & 1 & -2 \end{bmatrix}$$

$=$ circulant matrix

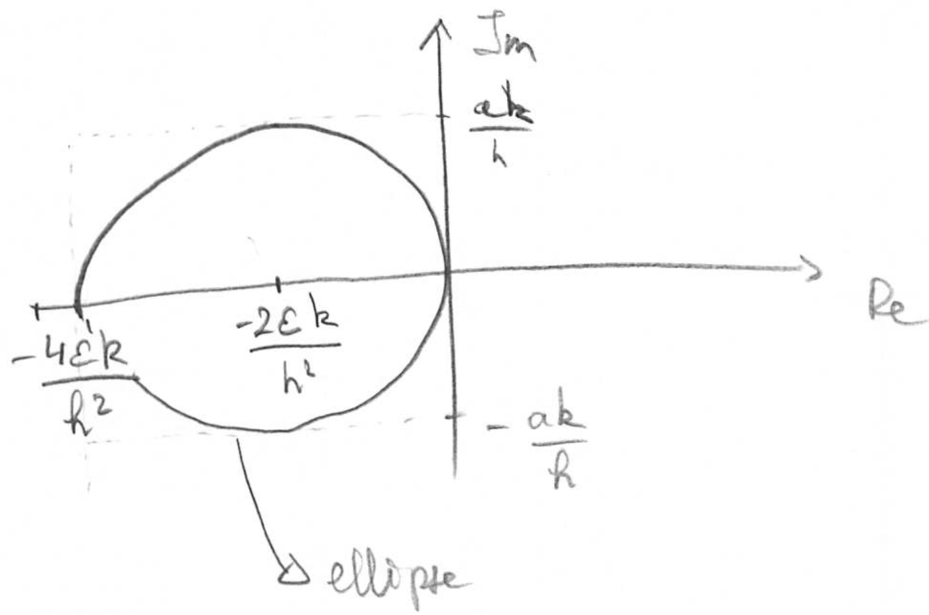
Since Laplacian is also a circulant matrix it has same eigenvectors as A. Therefore

$$\lambda_p(A_\varepsilon) = \lambda_p(A) + \varepsilon\lambda_p(\text{Laplacian})$$
$$= \underbrace{-\frac{ia}{h}\sin(2\pi ph)}_{\text{diff. op. part}} \underbrace{-\frac{2\varepsilon}{h^2}\left(1-\cos(2\pi ph)\right)}_{\text{Laplacian part}}$$

⚠ in general we <u>do</u> <u>not</u> have $\lambda_p(A+B) = \lambda_p(A) + \lambda_p(B)$

The only way this can happen is when A and B are diagonalizable in same basis of eigenvectors.

Plotting $k\lambda_p(A_\varepsilon)$ in complex plane:



For L.F. $\varepsilon = \frac{h^2}{2k}$ which means horizontal axis of ellipse is $[-2,0] \subset$ abs. stab. region for Euler.

If in addition we assume CFL i.e. $\left|\frac{ak}{h}\right| < 1$, then also vertical axis of ellipse $\subset$ abs. stab region for Euler.

$\Rightarrow$ method is stable (provided CFL is satisfied)

Lax-Wendroff method

Idea: choose ε s.t. method is 2nd order accurate in both time and space.
(like Leapfrog but with one step only and + dissipation)
(more commonly used)

$$U' = AU$$
$$U'' = AU' = AAU = A^2U$$

Then discretize system using Taylor's method:

$$U^{n+1} = U^n + kU' + \frac{k^2}{2}U''$$
$$= U^n + kAU^n + \frac{k^2}{2}A^2U^n$$

Computing A^2 gives:

$$\boxed{U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{a^2k^2}{8h^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)}$$

Upwind methods

If we wanted to use one sided differences in x:

$$u_x(x_j, t) \approx \frac{1}{h}\left(U_j - U_{j-1}\right)$$

$$u_x(x_j, t) \approx \frac{1}{h}\left(U_{j+1} - U_j\right)$$

we get two different methods:

① $$U_j^{n+1} = U_j^n - \frac{ak}{h}\left(U_j^n - U_{j-1}^n\right)$$ (first order in space and time)

② $$U_j^{n+1} = U_j^n - \frac{ak}{h}\left(U_{j+1}^n - U_j^n\right)$$

Which method should we use?

It depends on sign of a !!

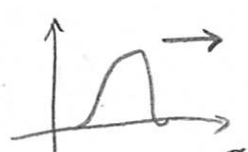
True solution satisfies:

$$u(x_j, t+k) = u(x_j - ak, t)$$

i.e. origin is shifted to ak after one time step k.

So when we advance in time:

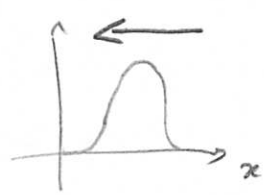
- $\underline{a>0}$



take values on left of U_j to preserve <u>causality</u>.

$\rightsquigarrow$ use method ①

- $\underline{a<0}$



take values on right of U_j to preserve <u>causality</u>.

$\rightsquigarrow$ use method ②

Stability analysis of upwind: we already did it!

Simply reuse arguments we used for L.F. since: upwind can be rewritten

$\underline{a>0}$: $$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{ak}{2h}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

$\underline{a<0}$ $$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) \underbrace{- \frac{ak}{2h}}_{= \frac{\varepsilon k}{h^2}}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

$$\Rightarrow \boxed{\varepsilon = \frac{|a|h}{2}}$$

For stability we need (i.e. for ellipse being inside abs. stab. region for Euler)

$$\underbrace{\left|\frac{ak}{h}\right| < 1}_{\text{CFL}} \quad \text{and} \quad -2 < -\underbrace{\frac{2\varepsilon k}{h^2}}_{\frac{|a|k}{h}} < 0$$

Note: getting signs wrong will cause blow up because inequality above is violated. (ellipse is on wrong side of imaginary axis)

Summary

Method	x accuracy	t accuracy	ε
Lax-Friedrichs	2	1	$h^2/2k$
Lax Wendroff	2	2	$a^2 k/2$
Euler	2	1	0
Upwind	1	1	$ah/2$
Leapfrog	2	2	N/A

come from $U' = A_\varepsilon U$ + Euler