MATH 5620 HW 2 Solutions.

problem 1 (B&F 5.1.4 a, b)

a)
$$\begin{cases} y' = e^{t-y} = f(t,y) & 0 \le t \le 1 \\ y(0) = 1 \end{cases}$$

i) $f(t,y)$ does not satisfy a Lipschitz condition on $D = \{(t,y) \mid 0 \le t \le 1,\ y \in \mathbb{R}\}$:

let $L > 0$ then: there is some $y_0 > 0$ for which:

$$|e^t - e^{t-y}| = e^t |1 - e^{-y}| > L|y|$$

when $y \le -y_0 < 0$.

This is because for $y < 0$:

$$|1 - e^{-y}| = e^{|y|} - 1 \geq |y| + \frac{|y|^2}{2} \geq L|y|$$

when $|y| \geq y_0 = \max(2(L-1), 0)$.

ii) Theorem 5.6 is inconclusive for well posedness: f is not Lipschitz in y var on D.

b)
$$\begin{cases} y' = \dfrac{1+y}{1+t} = f(t,y), & 0 \le t \le 1 \\ y(0) = 1 \end{cases}$$

i) $f(t,y)$ satisfies Lipschitz cond. on D because of Theorem 5.3 and:

$$\left|\frac{\partial f}{\partial y}(t,y)\right| = \left|\frac{1}{1+t}\right| \le 1 \quad \text{for all } (t,y) \in D.$$

ii) Theorem 5.6 guarantees this problem is well-posed.

Problem 2

$$\begin{cases} y' = \sqrt{y} \\ y(0) = 0 \end{cases}$$

Euler's method with $h > 0$:

$$y_0 = 0$$
$$y_1 = y_0 + h\sqrt{y_0} = 0$$
$$y_2 = y_1 + h\sqrt{y_1} = 0$$

$\rightsquigarrow$ $y_{i+1} = y_i + h\sqrt{y_i} = 0$

This is not surprising because problem has several solutions (and thus is not well posed):

$y(t) = t^2/4$ and $y(t) = 0$ solve problem, Euler's method picks only one.

```
>> drva
  t_i       Euler     Taylor2   Mod Euler        Heun         RK4        true
-----------------------------------------------------------------------------------
---
  0.5  0.00000000  0.12500000  0.56021113  0.27108851  0.29699746  0.28361652
    1  1.12042227  2.02323897  5.30148980  3.13272546  3.31431178  3.21909932
>> drvb
  t_i       Euler     Taylor2   Mod Euler        Heun         RK4        true
-----------------------------------------------------------------------------------
---
  2.5  2.00000000  1.75000000  1.81250000  1.84648277  1.83332336  1.83333333
    3  2.62500000  2.42578125  2.48155308  2.50941231  2.49997119  2.50000000
>> rkfdrv
normal termination
  i        t_i   y_i true        y_i        h_i
------------------------------------------------------------------------
  1  1.1101946  1.0051237  1.0051237  0.1101946
  2  1.2191314  1.0175211  1.0175212  0.1089368
  3  1.3572694  1.0396749  1.0396749  0.1381381
  4  1.5290112  1.0732756  1.0732757  0.1717417
  5  1.7470584  1.1213947  1.1213948  0.2180472
  6  2.0286416  1.1881700  1.1881702  0.2815832
  7  2.3994350  1.2795395  1.2795396  0.3707934
  8  2.8985147  1.4041842  1.4041843  0.4990798
  9  3.3985147  1.5285638  1.5285639  0.5000000
 10  3.8985147  1.6514962  1.6514963  0.5000000
 11  4.0000000  1.6762391  1.6762393  0.1014853
normal termination
  i        t_i   y_i true        y_i        h_i
------------------------------------------------------------------------
  1  1.1450265  0.1560234  0.1560235  0.1450265
  2  1.2822303  0.3254970  0.3254972  0.1372038
  3  1.4225752  0.5232641  0.5232644  0.1403449
  4  1.5482238  0.7234119  0.7234123  0.1256486
  5  1.6655759  0.9320282  0.9320288  0.1173521
  6  1.7773654  1.1521473  1.1521480  0.1117896
  7  1.8847226  1.3851226  1.3851234  0.1073571
  8  1.9881998  1.6316941  1.6316951  0.1034772
  9  2.0880997  1.8923290  1.8923303  0.0998999
 10  2.1846024  2.1673499  2.1673514  0.0965027
 11  2.2778244  2.4569932  2.4569949  0.0932220
 12  2.3678493  2.7614402  2.7614423  0.0900249
 13  2.4547442  3.0808350  3.0808374  0.0868949
 14  2.5385694  3.4152951  3.4152978  0.0838252
 15  2.6193834  3.7649189  3.7649220  0.0808140
 16  2.6972462  4.1297904  4.1297939  0.0778628
 17  2.7722206  4.5099829  4.5099869  0.0749744
 18  2.8443728  4.9055611  4.9055656  0.0721522
 19  2.9137726  5.3165830  5.3165880  0.0693998
 20  2.9804930  5.7431011  5.7431068  0.0667204
 21  3.0000000  5.8741000  5.8741059  0.0195070
>>
```



```
f = @(t,y) t*exp(3*t)-2*y;
df = @(t,y) 3*t*exp(3*t) + exp(3*t) - 2*f(t,y);
y = @(t) t*exp(3*t)/5 - exp(3*t)/25 + exp(-2*t)/25;
a=0; b=1; n=2; y0=0; h=(b-a)/n;

y1 = euler(a,b,n,y0,f);
y2 = taylor2(a,b,n,y0,f,df);
y3 = meuler(a,b,n,y0,f);
y4 = heun(a,b,n,y0,f);
y5 = rk4(a,b,n,y0,f);

fprintf('%5s %11s %11s %11s %11s %11s %11s\n','t_i','Euler','Taylor2','Mod Euler
','Heun','RK4','true');
fprintf('-----------------------------------------------------------------------
------------\n');
for i=1:n,
 ti=a+i*h;
 fprintf('%5.7g %11.8f %11.8f %11.8f %11.8f %11.8f %11.8f\n',ti,y1(i+1),y2(i+1),
y3(i+1),y4(i+1),y5(i+1),y(ti));
end;
```

```
f = @(t,y) 1+y/t;
df = @(t,y) f(t,y)/t - y/t^2;
y = @(t) t*log(t)+2*t;
a=1; b=2; n=4; y0=2; h=(b-a)/n;

y1 = euler(a,b,n,y0,f);
y2 = taylor2(a,b,n,y0,f,df);
y3 = meuler(a,b,n,y0,f);
y4 = heun(a,b,n,y0,f);
y5 = rk4(a,b,n,y0,f);

fprintf('%5s %11s %11s %11s %11s %11s %11s\n','t_i','Euler','Taylor2','Mod Euler
','Heun','RK4','true');
fprintf('-----------------------------------------------------------------------
------------\n');
for i=1:n,
 ti=a+i*h;
 fprintf('%5.7g %11.8f %11.8f %11.8f %11.8f %11.8f %11.8f\n',ti,y1(i+1),y2(i+1),
y3(i+1),y4(i+1),y5(i+1),y(ti));
end;
```



```
function y = euler(a,b,n,y0,f)
 y(1) = y0;
 h=(b-a)/n;
 for i=1:n,
  ti=a+(i-1)*h;
  y(i+1) = y(i) + h*f(ti,y(i));
 end;
```

```
% note: nobody agrees on what Heun's method is
% I am using def in B&F 5.4 9th ed.
function y = heun(a,b,n,y0,f)
 y(1) = y0;
 h=(b-a)/n;
 for i=1:n,
  ti=a+(i-1)*h;
  F1 = h*f(ti,y(i));
  F2 = h*f(ti+(1/3)*h,y(i)+(1/3)*F1);
  F3 = h*f(ti+(2/3)*h,y(i)+(2/3)*F2);
  y(i+1) = y(i) + (F1 + 3*F3)/4;
 end;
```



```
function y = meuler(a,b,n,y0,f)
 y(1) = y0;
 h=(b-a)/n;
 for i=1:n,
  ti=a+(i-1)*h;
  F1 = h*f(ti,y(i));
  F2 = h*f(ti+h,y(i)+F1);
  y(i+1) = y(i) + (F1 + F2)/2;
 end;
```

```
function y = rk4(a,b,n,y0,f)
 y(1) = y0;
 h=(b-a)/n;
 for i=1:n,
  ti=a+(i-1)*h;
  F1 = h*f(ti,y(i));
  F2 = h*f(ti+h/2,y(i)+F1/2);
  F3 = h*f(ti+h/2,y(i)+F2/2);
  F4 = h*f(ti+h  ,y(i)+F3);
  y(i+1) = y(i) + (F1 + 2*F2 + 2*F3 + F4)/6;
 end;
```



```
function y = taylor2(a,b,n,y0,f,df)
 y(1) = y0;
 h=(b-a)/n;
 for i=1:n,
  ti=a+(i-1)*h;
  y(i+1) = y(i) + h*f(ti,y(i)) + (h^2/2)*df(ti,y(i));
 end;
```

```
% Runge-Kutta-Fehlberg algorithm
% see Burden and Faires, Algorithm 5.3
function [y,t,hs]=rkf(a,b,y0,f,tol,hmax,hmin)
h=hmax; hs(1)=h; t(1)=a; y(1)=y0; flag=1; i=1;
while(flag==1)
 F1 = h*f(t(i)           ,y(i));
 F2 = h*f(t(i)+  h/4     ,y(i) +   F1/4);
 F3 = h*f(t(i)+3*h/8     ,y(i) + 3*F1/32 + 9*F2/32);
 F4 = h*f(t(i)+12*h/13  ,y(i) + 1932*F1/2197 - 7200*F2/2197 + 7296*F3/2197);
 F5 = h*f(t(i)+h         ,y(i) + 439*F1/216 - 8*F2 + 3680*F3/513 - 845*F4/4104);
 F6 = h*f(t(i)+h/2       ,y(i) -8*F1/27 + 2*F2 - 3544*F3/2565 + 1859*F4/4104 - 11
*F5/40);
 R = abs(F1/360 - 128*F3/4275 - 2197*F4/75240 + F5/50 + 2*F6/55)/h;
 if (R<=tol)
  t(i+1) = t(i) + h;
  y(i+1) = y(i) + 25*F1/216 + 1408*F3/2565 + 2197*F4/4104 - F5/5;
  hs(i+1) = h;
  i=i+1;
 end;
 delta = min(max(0.84*(tol/R)^(1/4),0.1),4);
 h=min(delta*h,hmax);
 if (t(i)>=b)
  flag=0;
 elseif (t(i)+h>b)
  h=b-t(i);
 elseif (h<hmin)
  flag=0;
  fprintf('abnormal termination: minimum h exceeded\n');
  return;
 end;
end;
fprintf('normal termination\n');
```



```
% Runge Kutta Fehlberg driver
%Check our Algorithm with book's example
%hmax=0.25; hmin=0.01; tol=1e-5;
%f = @(t,y) y - t^2 + 1; a=0; b=2; y0=1/2;
%y = @(t) (t+1)^2-0.5*exp(t);
%[ya,ta,ha]=rkf(a,b,y0,f,tol,hmax,hmin);
%for i=2:length(ta),
%fprintf('%3d %10.7f %10.7f %10.7f %10.7f\n',i-1,ta(i),y(ta(i)),ya(i),ha(i))
%end;

hmax=0.5; hmin=0.05; tol=1e-6;
f = @(t,y) y/t - (y/t)^2; a=1;b=4; y0=1;
y = @(t) t/(1+log(t));
[ya,ta,ha]=rkf(a,b,y0,f,tol,hmax,hmin);
fprintf('%3s %10s %10s %10s %10s\n','i','t_i','y_i true','y_i','h_i');
fprintf('------------------------------------------------------------------------
\n');
for i=2:length(ta),
fprintf('%3d %10.7f %10.7f %10.7f %10.7f\n',i-1,ta(i),y(ta(i)),ya(i),ha(i))
end;

f = @(t,y) 1+y/t + (y/t)^2; a=1;b=3; y0=0;
y = @(t) t*tan(log(t));
[ya,ta,ha]=rkf(a,b,y0,f,tol,hmax,hmin);
fprintf('%3s %10s %10s %10s %10s\n','i','t_i','y_i true','y_i','h_i');
fprintf('------------------------------------------------------------------------
\n');
for i=2:length(ta),
fprintf('%3d %10.7f %10.7f %10.7f %10.7f\n',i-1,ta(i),y(ta(i)),ya(i),ha(i))
end;
```