MATH 5620 practice Final

Problem 1 $y_n - 2y_{n-1} + y_{n-2} = h(f_n - f_{n-1})$

(a) This is an implicit method, as y_n appears in RHS

(b) The poly associated with the method are:

$$\rho(z) = z^2 - 2z + 1 \qquad q(z) = z^2 - z$$

$$\rho'(z) = 2z - 2$$

$$\left.\begin{array}{l} \rho(1) = 1 - 2 + 1 = 0 \\ \rho'(1) = 0 = q(1) \end{array}\right\} \Rightarrow \text{method is } \underline{\text{consistent}}$$

To check stability we need roots of $\rho(z)$:

$$\rho(z) = (z-1)^2 \Rightarrow \text{root } 1 \text{ has mult } 2.$$

$\Rightarrow$ method is <u>not stable</u> because there are roots w/ mult > 1 and $|z| = 1$.

$\Rightarrow$ method is not <u>convergent</u>.

Problem 2 A is assumed symmetric: $(A = A^T)$

Abbreviation:
ev: eigenvector
ew: eigenvalue

(a)

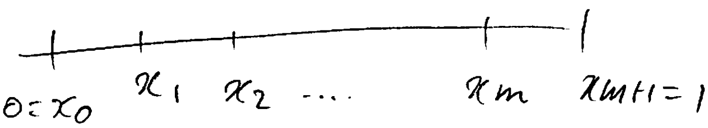
Power method	QR Algo
$v^{(0)}$ = some vector w/ $\|v^{(0)}\| = 1$	$A^{(0)} = Q_0^* A Q_0 = (\ddots)$ (red to tridiag. form)
for $k = 1, 2, \ldots$	for $k = 1, 2, \ldots$
$w^{(k)} = A v^{(k-1)}$	$Q^{(k)} R^{(k)} = A^{(k-1)}$
$v^{(k)} = w^{(k)} / \|w^{(k)}\|_2$	$A^{(k)} = R^{(k)} Q^{(k)}$
$\lambda^{(k)} = v^{(k)T} A v^{(k)}$	eigenvalues are in diag $(A^{(k)})$.

Fundamental difference:
- power method converges to largest eigenvalue (in magnitude)
- QR gives all eigenvalues of a matrix

	Power Method	QR Algo
Pro's	• does not need A explicitly, only how to apply A • cheap for large matrices • quadratic convergence to ew.	• gives all ew at once • convergence can be accelerated by using shifts to get cubic convergence for one eigenvalue.
Cons	• only gives largest ew • can have slow convergence if largest ew (in magnitude) is not simple.	• Can be expensive (specially red to tridiag form. • needs to store whole matrix • choice of shifts?

Problem 3

(a) The method of lines simply means we discretize PDE first in space:



$$U_1'(t) = -\frac{a}{2h}\left(U_2(t) - \underbrace{U_0(t)}_{\substack{\shortparallel \\ U_{m+1}(t) \text{ by periodic B.C.}}}\right)$$

$$U_i'(t) = -\frac{a}{2h}\left(U_{i+1}(t) - U_{i-1}(t)\right), \quad i = 2 .. m$$

$$U_{m+1}'(t) = -\frac{a}{2h}\left(\underbrace{U_{m+2}(t)}_{\substack{\shortparallel \\ U_1(t) \text{ by periodic B.C.}}} - U_m(t)\right)$$

MOL gives system of ODEs:

$$U'(t) = AU(t)$$

corner! because we have periodic BC.

where
$$A = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & \ddots & \ddots & 1 \\ 1 & & & -1 & 0 \end{bmatrix}$$

and
$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_{m+1}(t) \end{bmatrix}$$

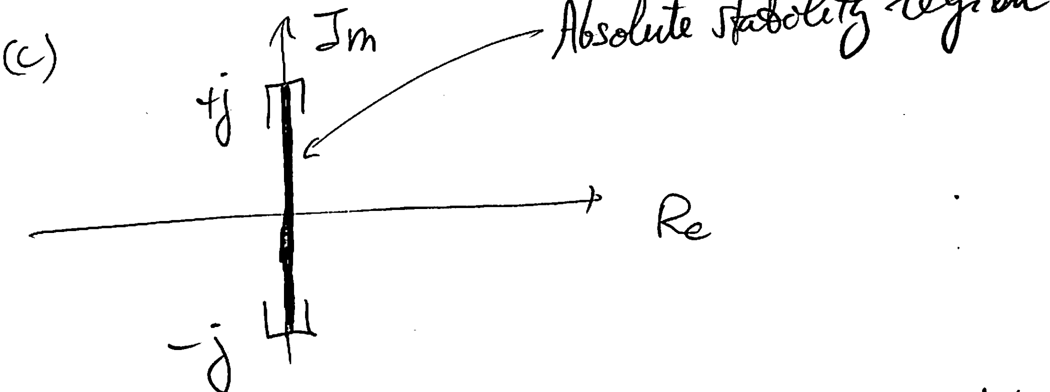
Problem 3 (Cont'd)

(b) Discretization in time gives

$$\frac{U^{n+1}-U^{n}}{2k} = AU^n$$

$$U^{n+1} = U^{n-1} + 2kAU^n$$

(c)



For stability of system of ODEs, we need to find k s.t.

$$k\lambda_p(A) \in \text{absolute stability region}, \; p=1,\dots,m+1.$$

Since $|\mathrm{Im}\,\lambda_p(A)| \le \frac{|a|}{h}$ ← absolute value important

in order to have

$$|\mathrm{Im}\,k\lambda_p(A)| \le 1 \text{ we need:}$$

$$\frac{k|a|}{h} \le 1 \iff \boxed{k \le \frac{h}{|a|}}$$

(CFL condition)

Problem 9

(a) Find $u_h \in V_h$ s.t.

$$a(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h.$$

(b) Galerkin $\perp$:

$$
\begin{array}{ll}
a(u, v_h) = (f, v_h) & (WF) \\
a(u_h, v_h) = (f, v_h) & (RG) \\
\hline
a(u - u_h, v_h) = 0 & \text{for all } v_h \in V_h.
\end{array}
$$

To show inequality:

$$\|u - u_h\|_E^2 = a(u - u_h, u - u_h)$$

$$\overset{\text{bilin of } a(\cdot,\cdot)}{=} a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h)$$

$$\overset{\text{Galerkin } \perp}{=} a(u - u_h, u - v_h) \quad \text{for all } v_h \in V_h.$$

$$\Rightarrow \boxed{\|u - u_h\|_E \leq \|u - v_h\|_E \quad \text{for all } v_h \in V_h.}$$

(c) Since $I_{\tau h} u \in V_h$:

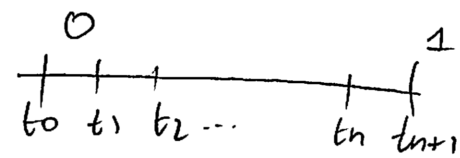
$$\|u - u_h\|_E \underset{(I1)}{\leq} \|u - I_{\tau h} u\|_E \underset{(I2)}{\leq} C \|u - I_{\tau h} u\|_{H^1(\Omega)}$$

$$\underset{(I_3)}{\leq} C h^k |u|_{H^{k+1}(\Omega)}$$

By (I2):

$$\|u - u_h\|_{H^1(\Omega)} \leq C h^k |u|_{H^{k+1}(\Omega)}.$$

Problem 5 $\begin{cases} y'' = f(t, y, y') & \text{for } t \in [0,1] \\ y(0) = \alpha \\ y(1) = \beta \end{cases}$



$t_i = ih, \; i = 0, \ldots n+1$

$h = \frac{1}{n+1}$

(a) The finite diff. approx to BVP can be written as a NL sys. of eq. w/ n unknowns:

$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ (the values y_0, y_{n+1} are specified by BC).

$E(\underline{y}) = \underline{0}$ where: $F: \mathbb{R}^n \to \mathbb{R}^n$

$\underline{y} \to \begin{bmatrix} F_1(\underline{y}) \\ \vdots \\ F_n(\underline{y}) \end{bmatrix}$

$$F_1(\underline{y}) = \frac{y_2 - 2y_1 + \overset{\alpha}{\cancel{y_0}}}{h^2} - f\left(t_1, y_1, \frac{y_2 - \overset{\alpha}{\cancel{y_0}}}{2h}\right)$$

$$F_i(\underline{y}) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right), \; i = 2 \ldots n-1$$

$$F_n(\underline{y}) = \frac{\overset{\beta}{\cancel{y_{n+1}}} - 2y_n + y_{n-1}}{h^2} - f\left(t_n, y_n, \frac{\overset{\beta}{\cancel{y_{n+1}}} - y_{n-1}}{2h}\right).$$

(b) We can write Jacobian $DF[\underline{y}]$ in the form:

$$DF[\underline{y}] = A - G_1 - G_2$$

where:

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} \in \mathbb{R}^{n \times n},$$

$$G_1 = \operatorname{diag}\left(f_y\left(t_1, y_1, \frac{y_2 - \alpha}{2h}\right), \dots, f_y\left(t_n, y_n, \frac{\beta - y_{n-1}}{2h}\right)\right)$$

$$G_2 = \frac{1}{2h}\begin{bmatrix} 0 & f_{y'}^{(1)} & & & \\ -f_{y'}^{(2)} & 0 & f_{y'}^{(2)} & & \\ & -f_{y'}^{(3)} & 0 & f_{y'}^{(3)} & \\ & & \ddots & \ddots & f_{y'}^{(n-1)} \\ & & & -f_{y'}^{(n)} & 0 \end{bmatrix}, \quad f_{y'}^{(i)} = f_{y'}\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$$

(c) $DF[\underline{y}]$ is a tridiag matrix $\Rightarrow$ one step of NM. for $\underline{F}(\underline{y}) = \underline{0}$ costs the same as solving a linear BVP on same grid.

(d) Typically one would take for $\underline{y}^{(0)}$ the values of line satisfying B.C.:

$$y_i^{(0)} = \alpha + ih(\beta - \alpha), \quad i = 0..n+1.$$

(check: $y_0^{(0)} = \alpha$ and $y_{n+1}^{(0)} = \beta$)

Problem 6

(a) $$Q^{-1}A = \begin{pmatrix} a_{11}^{-1} & & \\ & \ddots & \\ & & a_{nn}^{-1} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{bmatrix} 1 & a_{12}/a_{11} & \cdots & a_{1n}/a_{11} \\ \vdots & & & \\ a_{n1}/a_{nn} & a_{n2}/a_{nn} & \cdots & 1 \end{bmatrix}$$

$$I - Q^{-1}A = \begin{bmatrix} 0 & -a_{12}/a_{11} & \cdots & -a_{1n}/a_{11} \\ \vdots & & \ddots & \\ -a_{n1}/a_{nn} & -a_{n2}/a_{nn} & \cdots & 0 \end{bmatrix}$$

(b) $$\|I - Q^{-1}A\|_\infty = \max_{1 \le i \le n} \underbrace{\| e_i^T (I - Q^{-1}A) \|_1}_{\ell_1 \text{ norm of } i\text{-th row}}$$

$$= \max_{1 \le i \le n} \sum_{\substack{j=1 \\ j \ne i}}^{n} | a_{ij} / a_{ii} |$$

Condition is:

$$\|I - Q^{-1}A\|_\infty < 1 \iff \text{for all } i \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| < |a_{ii}|$$

$\iff$ strict diag dominance