

**Math 5620 – Numerical Analysis II**  
**Practice Final Exam (Spring 2012)**

**Problem 1.** Consider the multistep method:

$$y_n - 2y_{n-1} + y_{n-2} = h(f_n - f_{n-1}).$$

- (a) Is this method implicit or explicit?
- (b) Is this method convergent, stable and/or consistent? Justify your answer.

**Problem 2.**

- (a) Write pseudocode for the Power Method and the (unshifted) QR method for symmetric matrices. Please do not worry about stopping criteria.
- (b) Explain in your own words the differences, advantages and disadvantages of these two methods.

**Problem 3.** Consider the following advection equation (a hyperbolic problem) with periodic boundary conditions,

$$\begin{cases} u_t + au_x = 0, & t > 0, \quad 0 < x < 1, \\ u(0, t) = u(1, t), & t > 0, \\ u(x, 0) = \eta(x), & 0 < x < 1. \end{cases}$$

- (a) The interval  $[0, 1]$  is discretized with the points  $x_i = ih$ ,  $i = 0, \dots, m+1$ , where  $h = 1/(m+1) = \Delta x$ . Let  $U_i(t) \approx u(x_i, t)$ . Use the method of lines to approximate the PDE by a system of ODEs  $U'(t) = AU(t)$ , where  $U = [U_1, U_2, \dots, U_{m+1}]^T$  and  $A$  comes from the usual centered differences approximation to  $u_x$ .
- (b) Discretize in time using the Midpoint method. For a one dimensional IVP  $u' = f(u, t)$ , the Midpoint method is

$$\frac{u_{n+1} - u_{n-1}}{2k} = f(u_n, t_n).$$

Please use the notation  $U_i^n \approx u(x_i, t_n)$  (or  $U^n = [U_1^n, U_2^n, \dots, U_m^n]^T$  in vector form), where  $t_n = nk$  and  $k = \Delta t$  is the time step.

- (c) The absolute stability region for the Midpoint method is the segment between  $-j$  and  $j$  in the complex plane (here  $j = \sqrt{-1}$ ). Recall that eigenvalues of  $A$  are:

$$\lambda_p(A) = -\frac{aj}{h} \sin(2\pi ph), \quad p = 1, \dots, m+1.$$

For a given  $h$ , find a condition on  $k$  for the system of ODEs to be stable.

**Problem 4.** Let  $V$  be an appropriate subspace of functions defined on a domain  $\Omega$  and assume the weak formulation of a PDE gives the problem

$$\text{Find } u \in V \text{ such that } a(u, v) = (f, v) \text{ for all } v \in V,$$

here  $a(u, v)$  is a bilinear form,  $(f, v)$  is an inner product and  $V$  is an appropriate subspace of functions.

- (a) Let  $V_h$  be a finite dimensional subspace of  $V$ . Write down the Ritz-Galerkin formulation of the problem.
- (b) Let  $u$  be the solution to the weak formulation of the PDE and  $u_h$  be the solution to the Ritz-Galerkin problem. Show the basic error estimate (I1) below.
- (c) Using the inequalities (I1)–(I4) below, show the error estimate:

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{k+1}|u|_{H^{k+1}(\Omega)}.$$

(I1) *Ritz-Galerkin basic error estimate:*

$$\|u - u_h\|_E \leq \|u - v_h\|_E, \text{ for all } v_h \in V_h.$$

(I2) *Cea's Lemma:*

$$c\|u - u_h\|_{H^1(\Omega)} \leq \|u - u_h\|_E \leq C\|u - u_h\|_{H^1(\Omega)}.$$

(I3) *Interpolation error estimate:* Let  $I_{\tau_h}u$  be the interpolant of  $u$  on the triangulation  $\tau_h$ . If  $P_k$  (polynomials of degree up to  $k$ ) are used inside each element

$$\|u - I_{\tau_h}u\|_{H^1(\Omega)} \leq Ch^k|u|_{H^{k+1}(\Omega)}.$$

(I4) *Duality argument:*

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch\|u - u_h\|_{H^1(\Omega)}.$$

**Note:** Here  $c$  and  $C$  denote a generic positive constants. You do not need to keep track of such constants when deriving your error, also the particular definitions of the norms and seminorms are not needed in this problem.

**Problem 5.** Assume the non-linear BVP

$$\begin{cases} y'' = f(t, y, y'), & \text{for } t \in [0, 1], \\ y(0) = \alpha \\ y(1) = \beta, \end{cases}$$

has a unique solution. Here  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a smooth function. Let  $t_i = ih$ ,  $i = 0, \dots, n+1$ ,  $h = 1/(n+1)$ . Recall the finite difference approximations

$$\begin{aligned} y'(t_i) &= \frac{1}{2h}(y(t_{i+1}) - y(t_{i-1})) + \mathcal{O}(h^2) \\ y''(t_i) &= \frac{1}{h^2}(y(t_{i+1}) - 2y(t_i) + y(t_{i-1})) + \mathcal{O}(h^2). \end{aligned}$$

- (a) Let  $y_i \approx y(t_i)$ . Use the finite difference method to write the approximation to the non-linear BVP in the form

$$\mathbf{F}(\mathbf{y}) = \mathbf{0},$$

with  $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\mathbf{F}(\mathbf{y}) = (F_1(\mathbf{y}), \dots, F_n(\mathbf{y}))^T$  and  $\mathbf{y} = (y_1, \dots, y_n)^T$ . Please be particularly careful with the equations corresponding to the end of the interval.

- (b) Recall Newton's method for solving  $\mathbf{F}(\mathbf{y}) = \mathbf{0}$ :

**Let  $\mathbf{y}^{(0)}$  be an initial guess**  
**for**  $k = 1, \dots$   
 $\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} - (D\mathbf{F}[\mathbf{y}^{(k)}])^{-1}\mathbf{F}(\mathbf{y}^{(k)})$   
**end**

Find the Jacobian  $D\mathbf{F}[\mathbf{y}^{(k)}] = (\nabla F_1(\mathbf{y}^{(k)}), \dots, \nabla F_n(\mathbf{y}^{(k)}))^T \in \mathbb{R}^{n \times n}$  of  $\mathbf{F}$ .

- (c) How does the computational cost of one Newton's method step compares to solving a linear BVP on the same grid?  
 (d) What can be used as initial guess  $\mathbf{y}^{(0)}$ ?

**Problem 6.** Let  $A$  be a  $n \times n$  matrix and  $Q = \text{diag}(a_{11}, \dots, a_{nn})$ .

- (a) Compute  $Q^{-1}A$  and  $I - Q^{-1}A$ .  
 (b) Show that

$$\|I - Q^{-1}A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1, j \neq i}^n \frac{|a_{ij}|}{|a_{ii}|}.$$

(recall the induced matrix norm  $\|B\|_\infty$  is the largest 1-norm of the rows of  $B$ )

- (c) Recall the Jacobi method for solving the linear system  $Ax = b$ ,

$$Qx^{(k+1)} = (Q - A)x^{(k)} + b,$$

with  $Q$  given as above. Based on part (b), give a condition on relative size of the diagonal and off-diagonal elements of  $A$  that guarantees convergence of Jacobi method.