MATH 5620 - Numerical Analysis
Practice Midterm

Problem 1 $\quad y_n - y_{n-2} = \frac{h}{3}\left[f_n - 3f_{n-1} + 2f_{n-2}\right]$

(a) This is an implicit method since $f_n = f(t_n, y_n)$

(b) The associated polynomials to this method are:

$p(z) = z^2 - 1$

$q(z) = \frac{1}{3}\left[z^2 - 3z + 2\right]$

- This method is stable since the roots of $p(z)$ are ± 1 (simple roots w/ $|z|=1$)

- This method is not consistent since:

$p(1) = 0$ ✓

$p'(z) = 2z$

$p'(1) = 2$

$q(1) = \frac{1}{3}\left[1 - 3 + 2\right] = 0$

$p'(1) \neq q(1)$

Hence method is not convergent, since convergent $\Leftrightarrow$ (stable & consistent)

Problem 2

(a) Jacobi method for solving $Ax = b$

$x^{(0)}$ = given

For $k = 1 \ldots$

$\quad x^{(k)} = D^{-1}\left[(D-A)x^{(k-1)} + b\right]$

where D = diagonal of A = diag(diag(A))

(b) $$D^{-1}A = \begin{bmatrix} 1 & a_{12}/a_{11} & \cdots & a_{1n}/a_{11} \\ a_{21}/a_{22} & 1 & \cdots & a_{2n}/a_{22} \\ \vdots & & & \vdots \\ a_{n1}/a_{nn} & \cdots & & 1 \end{bmatrix}$$

$$I - D^{-1}A = -\begin{bmatrix} 0 & a_{12}/a_{11} & \cdots & a_{1n}/a_{11} \\ a_{21}/a_{22} & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n1}/a_{nn} & \cdots & a_{nn-1}/a_{nn} & 0 \end{bmatrix}$$

$$\|I - D^{-1}A\|_\infty = \max_{i=1..n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|}$$

Since A is diag. dom.: $$\frac{\sum_{j=1, j\neq i}^{n} |a_{ij}|}{|a_{ii}|} < 1$$

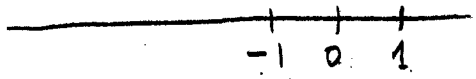
$\Rightarrow \|I - D^{-1}A\|_\infty < 1$

(c) A matrix splitting method converges when $\|I - Q^{-1}A\| < 1$ for some induced matrix norm. Here $Q = D$, so the method converges for diag. dom. matrices.

(d) This is only a sufficient condition for convergence. There may be matrices that are not diag. dom and for which Jacobi converges.

Problem 3

(a) We need to devise an integration formula that is exact for polynomials up to degree 2:



$$\int_0^1 p(t)\,dt = A\,p(1) + B\,p(0)$$

Using basis $p_0(t) = 1$
$p_1(t) = t$

we get system:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \Rightarrow B = \tfrac{1}{2},\ A = \tfrac{1}{2}$$

Thus we get A-M:

$$\boxed{y_{n+1} = y_n + \frac{h}{2}\left[f_{n+1} + f_n\right]}$$ (implicit trapezoidal rule)

(b) Written in the general form our method's coefficients are:

$a_1 = 1$ $\quad$ $b_1 = \frac{1}{2}$
$a_0 = -1$ $\quad$ $b_0 = \frac{1}{2}$

therefore:

$$d_0 = a_1 + a_0 = 0$$

$$d_1 = (0 \cdot a_0 - b_0) + \left(\frac{1}{1!}a_1 - \frac{1^0}{0!}b_1\right)$$

$$= -\frac{1}{2} + 1 - \frac{1}{2} = 0$$

$$d_2 = (0 \cdot a_0 - 0 b_0) + \frac{1^2}{2!}a_1 - \frac{1^2}{1!}b_1$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

$$d_3 = (0 \cdot a_0 - 0 b_0) + \frac{1^3}{3!}a_1 - \frac{1^2}{2!}b_1$$

$$= \frac{1}{6} - \frac{1}{4} = -\frac{1}{12} \neq 0$$

Therefore the method is of order 2 and the local truncation error is:

$$-\frac{1}{12} h^3 y^{(3)}(t_{n-1}) + O(h^4).$$

Problem 4

(a) $y(t+h) = y(t) + h y'(t) + \frac{h^2}{2} y''(t) + O(h^3)$

$= y + hf + h^2 \frac{df}{dt} + O(h^3)$

and $\frac{df(t,y)}{dt} = f_t(t,y) + y' f_y(t,y)$

$= f_t + f f_y$

Thus:

$$\boxed{y(t+h) = y + hf + \frac{h^2}{2}[f_t + f f_y] + O(h^3)}$$

(b) $y(t+h) = y + w_1 hf + w_2 hf(t+\alpha h, y+\beta hf) + O(h^3)$

$= y + w_1 hf + w_2 h\left[f + \alpha h f_t + \beta h f f_y + O(h^2)\right] + O(h^3)$

$= y + \underbrace{w_1 hf + w_2 hf}_{= h(w_1+w_2)f} + w_2 h^2(\alpha f_t + \beta f f_y) + O(h^3)$

(c) For the method to be order 2, it has to match Taylor series from (a):

$w_1 + w_2 = 1$ (h term)

$w_2 \alpha = \frac{1}{2}$ (h^2 term)

$w_2 \beta = \frac{1}{2}$

(d) If $w_1 = \frac{1}{4}$, $w_2 = \frac{3}{4}$ and: $\alpha = \frac{1}{2}\frac{4}{3} = \frac{2}{3}$, $\beta = \frac{1}{2}\frac{4}{3} = \frac{2}{3}$

Thus the method is:

y_0 = initial condition

for $i = 0 \ldots n$

$F_1 = h f(t_i, y_i)$

$F_2 = h f(t_i + \frac{2}{3}h, y_i + \frac{2}{3}F_1)$

$y_{i+1} = y_i + \frac{1}{4}F_1 + \frac{3}{4}F_2$