SOME LINEAR ALGEBRA FACTS

Here are some linear algebra facts that you should know for Math 5620. The most important ones will be reviewed in more detail in class (for example the different definitions of $\mathbf{A}$ invertible). Please also look at the class text book §6.3 and §6.4.

- Let $\mathbf{A}$ be a $n \times m$ matrix. It has n rows and m columns and its elements are often written:
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = [a_{ij}]_{i=1\ldots n, j=1\ldots m}.$$
- To say $\mathbf{A}$ is a real $n \times m$ matrix we also use the notation $\mathbf{A} \in \mathbb{R}^{n\times m}$.
- A vector is a $n \times 1$ matrix. For example $\mathbf{x} \in \mathbb{R}^n$ written entrywise
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$
- The transpose of a $n \times m$ matrix $\mathbf{A}$ is a $m \times n$ matrix defined as follows:
$$(\mathbf{A}^T)_{i,j} = (\mathbf{A})_{j,i}, \text{ for } i = 1, \ldots, n \text{ and } j = 1, \ldots, m.$$
- We use the notation $\mathbf{e}_j \in \mathbb{R}^n$ for the $j-$th canonical basis vector of $\mathbb{R}^n$, that is
$$\mathbf{e}_j = [0, \ldots, 0, \underbrace{1}_{j-\text{th position}}, 0, \ldots, 0]^T$$
- The canonical basis vectors are useful for example for picking up the $i, j-$th entry of a matrix:
$$a_{i,j} = \mathbf{e}_i^T \mathbf{A} \mathbf{e}_j^T.$$
- Let $\mathbf{A} \in \mathbb{R}^{n\times m}$, $x \in \mathbb{R}^m$. The matrix vector product $\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^n$ can be written componentwise:
$$(\mathbf{A}\mathbf{x})_i = \sum_{j=1}^{m} a_{ij} x_j.$$
The matrix vector product can also be written in terms of the columns. So denote by $a_i \in \mathbb{R}^n$ the $i-$th column of A, that is we partition A as:
$$\mathbf{A} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_m \\ | & | & \cdots & | \end{bmatrix}$$
Then
$$\mathbf{A}\mathbf{x} = \sum_{j=1}^{m} x_i \mathbf{a}_i$$

- For a $n \times n$ real matrix $\mathbf{A}$ we have
$$\begin{aligned} & \mathbf{A} \text{ invertible} \\ \Leftrightarrow\ & \det \mathbf{A} \neq 0 \\ \Leftrightarrow\ & \mathbf{A} \text{ non-singular} \\ \Leftrightarrow\ & \ker \mathbf{A} = \{0\} \\ \Leftrightarrow\ & \mathbf{A}\mathbf{x} = 0 \ \Rightarrow\ \mathbf{x} = 0 \\ \Leftrightarrow\ & \text{The columns of } \mathbf{A} \text{ are linearly independent} \\ \Leftrightarrow\ & \text{There exists a unique inverse } \mathbf{A}^{-1} \end{aligned}$$
- If $\mathbf{A} \in \mathbb{R}^{n\times p}$ and $\mathbf{B} \in \mathbb{R}^{p\times m}$ then the matrix-matrix product $\mathbf{AB} \in \mathbb{R}^{n\times m}$ is defined as follows
$$(\mathbf{AB})_{i,j} = \sum_{k=1}^{p} a_{ik} b_{kj}, \text{ for } i = 1, \ldots, n \text{ and } j = 1, \ldots, m.$$
- In general the matrix matrix product is not commutative. Before even considering commuting the dimensions have to agree so $\mathbf{A}$ and $\mathbf{B}$ must be square. And for square matrices in general $\mathbf{AB} \neq \mathbf{BA}$.
- The dot (or inner) product between $\mathbb{R}^n$ vectors $\mathbf{x}$ and $\mathbf{y}$ is the scalar
$$\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$
- The entry $(\mathbf{AB})_{ij}$ of the matrix product $\mathbf{AB}$ is the dot product between the $i-$th row of $\mathbf{A}$ and the $j-$th column of $\mathbf{B}$.