# MATH 5620 NUMERICAL ANALYSIS II
# HOMEWORK 5, DUE TUESDAY APRIL 12 2011

The goal of this homework is to implement 1D P2 finite elements for approximating the solution to the BVP

$$\begin{cases} -u'' = f \\ u(0) = u(1) = 0 \end{cases}$$

at the nodes $x_i = ih/2$, for $i = 0, \dots, 2n$ and $h = 1/n$. With this notation the computational domain $[0, 1]$ is divided in $n$ elements of length $h = 1/n$, which are the intervals $[x_{2(i-1)}, x_{2i}]$ for $i = 1, \cdots, n$.

(a) Let $\widehat{x}_i = i/2$, for $i = 0, 1, 2$. Write down the equations for the second degree polynomials $\widehat{\phi}_i(\widehat{x})$ in Lagrange form such that

$$\widehat{\phi}_i(\widehat{x}_j) = \delta_{ij}, \text{ for } i, j = 0, 1, 2.$$

(b) Compute the local stiffness matrix $\widehat{A}_{loc} \in \mathbb{R}^{3\times 3}$ for the reference element $[0, 1]$, with entries

$$(\widehat{A})_{i,j} = \int_0^1 \widehat{\phi}'_{i-1}(\widehat{x})\widehat{\phi}'_{j-1}(\widehat{x})d\widehat{x}, \text{ for } i, j = 1, 2, 3.$$

You may use a symbolic calculation or an integration rule that is exact for polynomials of degree 2 (e.g. Simpson's rule). Giving only numerical values is OK.

(c) Compute the local mass matrix $\widehat{M}_{loc} \in \mathbb{R}^{3\times 3}$ for the reference element $[0, 1]$, with entries

$$(\widehat{M})_{i,j} = \int_0^1 \widehat{\phi}_{i-1}(\widehat{x})\widehat{\phi}_{j-1}(\widehat{x})d\widehat{x}, \text{ for } i, j = 1, 2, 3.$$

Since these calculations are tedious to do by hand, I strongly recommend that you use a numerical integration rule exact for polynomials of degree 4 to evaluate these integrals and give numerical values. You may use for example the three point Gaussian quadrature:

$$\int_{-1}^1 f(t)dt \approx \frac{1}{9}\left[5f\left(\frac{-\sqrt{15}}{5}\right) + 8f(0) + 5f\left(\frac{\sqrt{15}}{5}\right)\right].$$

**Warning:** This quadrature is given for $[-1, 1]$ but the integrals you need to evaluate are on $[0, 1]$, so you need to adjust the weights and nodes of the quadrature accordingly.

(d) What is the local to global map? (The approximation space is continuous)

(e) Modify the Matlab code `rg.m` (class website) to solve the BVP with P2 elements instead of P1 elements and the same source term $f(x) = \exp(x)$. The true solution is $u(x) = -e^x + (e-1)x + 1$. Display **on the same plot** the true solution, the P1 solution with $n = 3$ and the P2 solution with $n = 3$.

**Warning:** You do need to plot the approximated solution in between the nodes. Here is a sample code to do this, using the same variables as in `rg.m`:

```
% grid definitions, true solution, source term etc...

% local basis functions in parent element [0,1]
phi{1} = @(x) SOME FUNCTION OF x;
phi{2} = @(x) SOME FUNCTION OF x;
phi{3} = @(x) SOME FUNCTION OF x;

% local basis functions derivatives in parent element [0,1]
dphi{1} = @(x) SOME FUNCTION OF x;
dphi{2} = @(x) SOME FUNCTION OF x;
dphi{3} = @(x) SOME FUNCTION OF x;

% CODE TO COMPUTE Ahat and Mhat

% MATRIX AND RHS ASSEMBLY

% solve
u = A\b;

% evaluate function in grid
up2 = zeros(size(xf));
for e=1:n,
 % indices of nodes in xf that are inside element e
 ii =  x(i(e,1)) <= xf & xf < x(i(e,3));
 for j=1:3,
  up2(ii) = up2(ii) ...
        + u(i(e,j))*phi{j}((xf(ii)    - x(i(e,1))) ...
                          /(x(i(e,3)) - x(i(e,1))));
 end;
end;

% plot and compare
plot(xf,up2,x,up1,xf,utrue(xf));
```

(f) **Extra credit:** Produce a `loglog` plot comparing the $L^2$ error of the P1 and P2 solutions for different values of $n$. The abscissa should be $h = 1/n$ and the ordinate the error $||u - u_S||$, with the $L^2$ norm. You may take for values of `n=[10,100,1000,5000]`. Give estimates of the convergence rate of both methods.