HW1 SOLUTIONS
MATH 5620 - S11

Problem 1: Take $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, $A = A^T$ and all entries are > 0.

However: $[1 \ -1] \, A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -4$

Problem 2 Find Cholesky factorization of

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$l_{11} = (a_{11})^{\frac{1}{2}} = 1$$
$$l_{21} = (a_{21} - 0)/l_{11} = -1$$
$$l_{22} = (a_{22} - l_{21}^2)^{\frac{1}{2}} = (2-1)^{\frac{1}{2}} = 1$$

$\Rightarrow \boxed{L = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}$ check $A = LL^T$

Problem 3 Let $A = \begin{bmatrix} \alpha & 1 \\ 1 & 1 \end{bmatrix}$

Then $K_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$

$$\boxed{\|A\|_\infty = \max\{|\alpha|+1, 2\} = \begin{cases} |\alpha|+1 & \text{if } |\alpha| \geq 1 \\ 2 & \text{if } |\alpha| < 1 \end{cases}}$$

$$A^{-1} = \frac{1}{\alpha - 1} \begin{bmatrix} 1 & -1 \\ -1 & \alpha \end{bmatrix}$$

$$\boxed{\Rightarrow \|A^{-1}\|_\infty = \max\left\{\frac{2}{|\alpha-1|}, \frac{1+|\alpha|}{|\alpha-1|}\right\} = \begin{cases} \frac{|\alpha|+1}{|\alpha-1|} & \text{if } |\alpha| \geq 1, \alpha \neq 1 \\ \frac{2}{|\alpha-1|} & \text{if } |\alpha| < 1 \end{cases}}$$

Therefore:

$$\mathcal{K}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = \begin{cases} \dfrac{(1+|\alpha|)^2}{|\alpha-1|} & |\alpha|>1 \\ \dfrac{4}{|\alpha-1|} & |\alpha|<1 \end{cases}$$

note that as $\alpha \to 1$, $\mathcal{K}_\infty(A) \to \infty$ (i.e. matrix becomes increasingly singular).

Let $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$, $A^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

$\Rightarrow \|A\|_\infty = 2 \qquad \|A^{-1}\|_\infty = 1$

$\Rightarrow \boxed{\mathcal{K}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 2}$

Problem 4 $\quad \mathcal{K}(\lambda A) = \|\lambda A\| \, \|(\lambda A)^{-1}\|$

$= |\lambda| \, \|A\| \, |\lambda|^{-1} \|A^{-1}\|$

$= \mathcal{K}(A)$ when $\lambda \neq 0$.

Problem 5 $\quad$ Show that if A is invertible and

$$\|B-A\| < \|A^{-1}\|^{-1}$$

then B is invertible.

Notice: $I - A^{-1}B = A^{-1}(A-B)$

$\|I - A^{-1}B\| = \|A^{-1}(A-B)\| \leq \|A^{-1}\| \, \|A-B\|$

Thus:

$\|B-A\| < \|A^{-1}\|^{-1} \Leftrightarrow \|A^{-1}\| \, \|A-B\| < 1 \Rightarrow \|I-A^{-1}B\| < 1$

Hence using Neumann series:

$$\|I - A^{-1}B\| < 1 \quad \text{guarantees}$$

i) $I - (I - A^{-1}B) = A^{-1}B$ is invertible

ii) $(A^{-1}B)^{-1} = B^{-1}A = \sum_{k=0}^{\infty} (I - A^{-1}B)^k$

$$\Rightarrow \boxed{B^{-1} = \sum_{k=0}^{\infty} (I - A^{-1}B)^k A^{-1}.}$$

problem 6: See attached code & output.

```
>> test_mylu_sol

A =

     0     2    -1
     1    -1     2
     1    -1     4


L =

     1     0     0
     0     1     0
     1     0     1


U =

     1    -1     2
     0     2    -1
     0     0     2


P =

     0     1     0
     1     0     0
     0     0     1

error=0

A =

     1     1    -1     2
    -1    -1     1     5
     2     2     3     7
     2     3     4     5


L =

    1.0000         0         0         0
    1.0000    1.0000         0         0
    0.5000         0    1.0000         0
   -0.5000         0   -1.0000    1.0000


U =

    2.0000    2.0000    3.0000    7.0000
         0    1.0000    1.0000   -2.0000
         0         0   -2.5000   -1.5000
         0         0         0    7.0000


P =

     0     0     1     0
     0     0     0     1
     1     0     0     0
     0     1     0     0

error=0
```



```
% B&F 6.5.4 a
A = [ 0  2 -1
      1 -1  2
      1 -1  4]
[L,U,P]=mylu(A)
fprintf('error=%g\n',norm(P*A - L*U,'inf'));

% B&F 6.5.4 c
A = [ 1  1 -1 2
     -1 -1  1 5
      2  2  3 7
      2  3  4 5]
[L,U,P]=mylu(A)
fprintf('error=%g\n',norm(P*A - L*U,'inf'));
```

```
function [L,U,P] = mylu(A)
n = size(A,1);
U=A; L=eye(n); P=eye(n);
for k=1:n-1,
 % choose pivot
 [val,i] = max(abs(U(k:n,k)));
 i = i+k-1;
 % switch k-th row with pivot row i
 U([i,k],k:n) = U([k,i],k:n);
 % switch rows in L
 L([i,k],1:k-1)=L([k,i],1:k-1);
 % switch rows in P
 P([i,k],:) = P([k,i],:);
 % update
 for j=k+1:n,
  L(j,k) = U(j,k)/U(k,k);
  U(j,k:n) = U(j,k:n) - L(j,k)*U(k,k:n);
 end;
end;
```