

## MATH 5620 NUMERICAL ANALYSIS II

### FINAL REVIEW SHEET

#### CHAPTER 6 – DIRECT METHODS FOR LINEAR SYSTEMS

- Know how to do Gaussian elimination on a simple matrix (say up to  $4 \times 4$ ). Partial pivoting. I could give an example similar to the ones we saw in class where things break down for Gaussian elimination but where the problems go away with pivoting.
- You do not need to remember the algorithm for the  $LU$  factorization, but you should know what the factorization means, what happens in the symmetric positive definite case (Cholesky). How much does it cost to compute  $L$  and  $U$ ? What is the work involved in the forward and backward substitution? Can you write the algorithm for forward and/or backward substitution in pseudocode?
- What is a symmetric positive definite matrix?

#### CHAPTER 7 – ITERATIVE METHODS FOR LINEAR SYSTEMS

- Neumann series, condition for convergence, spectral radius, induced matrix norm. Iterative refinement.
- Jacobi, Gauss-Seidel, SOR. Advantages, disadvantages and pseudocode.
- Condition number error bound (7.19) and interpretation. We did not see Theorem 7.29 and it is not included in the exam.
- Conjugate gradient: do not memorize algorithm. You should be able to explain what are the advantages and disadvantages of this method with respect to direct methods. Why is CG a good method for sparse matrices? What is a preconditioner?

#### CHAPTER 5 – INITIAL VALUE PROBLEMS

- §5.11 Stiff differential equations. A-stability and how to determine if a linear multi-step method is A-stable. What is the region of absolute stability and what does it mean? (both are linked to how the method fares with the simple problem  $y' = \lambda y; y(0) = 1$  with a given time step  $h$ ).

#### CHAPTER 11 – BOUNDARY VALUE PROBLEMS

- Theorem giving sufficient condition for the problem to have a unique solution.

- §11.1–11.2: Linear shooting method and Non-Linear Shooting method. You should know the basic ideas behind these two methods. Without knowing the formulas by heart you should be able to linearly combine two solutions to a linear IVP to get the solution to a linear BVP. Also you should understand how non-linear shooting works (with secant and Newton's method)
- §11.3–11.4: Finite Difference Methods. For linear BVP know how to construct the linear system. For Non-Linear BVP: know how to setup the non-linear system (Jacobian etc. . . ) and Newton's method.

## CHAPTER 12 – NUMERICAL SOLUTION TO PARTIAL DIFFERENTIAL EQUATIONS

**Note:** What we saw in class is significantly different from what is covered in the class textbook, so please refer to the class notes.

- **Elliptic problems:** (Laplace equation  $\Delta u = f$ ) Five point stencil for Laplacian and how the discretization works. Know that the local truncation error is  $\mathcal{O}(h^2)$  and how to derive it.
- **Parabolic problems:** (heat equation  $u_t = \kappa \Delta u$ ).
  - Method of lines discretization. Euler and implicit Trapezoidal rule (Crank-Nicholson).
  - Please do not remember the expressions for the eigenvalues of the discrete Laplacian. However you should be able to use these expressions to determine the region of stability of a discretization method: for example for Euler's method you should be able to get that  $k/h^2 < 1/2$  from the expression of the eigenvalues for the discrete Laplacian.
  - Know the idea of LOD and ADI for multidimensional problems. What does one gain by using these “operator splitting” methods?
- **Hyperbolic problems**
  - You need to understand Leapfrog, upwind, Lax-Friedrich and Lax-Wendroff method. It is not necessary to remember these methods and/or their derivations by heart.
  - For a given method you should be able to show where the eigenvalues of the iteration matrix lie (given the eigenvalues of the discrete Laplacian and those of the matrix representing the difference operator) and deduce the CFL condition from the stability conditions (i.e. the eigenvalues of the iteration matrix should be inside the absolute stability region of the method used for the time discretization).
- **Finite Elements** (only for Laplace equation  $-\Delta u = f$ )
  - Know weak formulation (1D and 2D)

- Ritz-Galerkin approximation, Galerkin orthogonality, basic error estimate (with the energy norm), optimality of the Ritz-Galerkin approximation in the energy norm.
- Know how to assemble the stiffness matrix and right hand side for 1D P1 elements (2 chain rules + change of integration variable).

## CHAPTER 9 – EIGENVALUES

**Note:** What we saw in class is significantly different from what is covered in the class textbook, so please refer to the class notes.

- Know how to write pseudocode for the power method, symmetric power method, inverse iteration and Rayleigh quotient iteration.
- You should be able to say what are the advantages/disadvantages of such methods from the computational point of view.
- Know the idea behind Householder reflectors and how they can be used to reduce a matrix to Hessenberg (or tridiagonal) form. You do not need to know how to choose the Householder reflector that is more numerically stable.
- You should know the basic version of the QR algorithm and the simultaneous iteration algorithm (which can be used to study the QR algorithm see p71 in math5620s11\_11.pdf). What is the computational cost of one iteration of the QR algorithm? What does it compute? Where are the eigenvalues and eigenvectors when the algorithm terminates?

## OTHER CONSIDERATIONS

- Understanding of the methods will be favored over simple memorization.
- You should be able to evaluate a method based on its practical merits (does it take too many function evaluations?...)
- Expect to write pseudo-code
- You are expected to know how to solve  $2 \times 2$  systems (and simple  $3 \times 3$  or triangular systems). Also you should know how to find the roots of a quadratic or a simple cubic or quartic polynomial.
- A calculator is not needed for the exam, and is not allowed.
- Two one-sided (or equivalent) letter sized, handwritten cheat sheet is allowed.
- No books or other notes are allowed.