MATH 5620 - SPRING 2011
PRACTICE FINAL SOLUTIONS

Problem 1

$$y_n - 2y_{n-1} + y_{n-2} = h(f_n - f_{n-1})$$

(a) This is an implicit method, as y_n appears in RHS.

(b) The polynomials associated with the method are.

$\rho(z) = z^2 - 2z + 1 \qquad q(z) = z^2 - z$

$\rho'(z) = 2z - 2$

$$\left.\begin{array}{l}\rho(1) = 1 - 2 + 1 = 0 \\ \rho'(1) = 0 = q(1)\end{array}\right\} \Rightarrow \text{method is consistent.}$$

To check stability, we need roots of $\rho(z)$

$$\rho(z) = (z-1)^2 \Rightarrow \text{root } 1 \text{ has multiplicity } 2.$$

$\Rightarrow$ method is not stable because there are roots with $|z|=1$ and multiplicity ≥ 1

$\Rightarrow$ method is not convergent.

Problem 2

(a)

$$Q^{-1}A = \begin{pmatrix} a_{11}^{-1} & & \\ & \ddots & \\ & & a_{nn}^{-1} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{bmatrix} 1 & a_{12}/a_{11} & \cdots & a_{1n}/a_{11} \\ \vdots & & & \\ a_{n1}/a_{nn} & a_{n2}/a_{nn} & \cdots & 1 \end{bmatrix}$$

$$I - Q^{-1}A = \begin{bmatrix} 0 & -a_{12}/a_{11} & \cdots & -a_{1n}/a_{11} \\ \vdots & & & \\ -a_{n1}/a_{nn} & -a_{n2}/a_{nn} & \cdots & 0 \end{bmatrix}$$

ℓ_1 norm of i-th row

(b) $$\|I - Q^{-1}A\|_\infty = \max_{1 \le i \le n} \|e_i^T (I - Q^{-1}A)\|_1$$

$$= \max_{1 \le i \le n} \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}/a_{ii}|$$

Condition is $\|I - Q^{-1}A\|_\infty < 1 \iff$ for all i: $\sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| < |a_{ii}|$ (diag. dominance)

__Problem 3__ A is assumed symmetric $(A = A^T)$

(a)

__Power method__

$v^{(0)}$ = some vector with $\|v^{(0)}\|_2 = 1$

for $k = 1, 2, \ldots$

$w^{(k)} = A v^{(k-1)}$

$v^{(k)} = w^{(k)} / \|w^{(k)}\|_2$

$\lambda^{(k)} = v^{(k)T} A v^{(k)}$

__QR Algorithm__

$A^{(0)} = Q_0^* A Q_0 = (\diagdown\!\diagdown\!\diagdown)$

(reduction to tridiagonal form)

for $k = 1, 2, \ldots$

$Q^{(k)} R^{(k)} = A^{(k-1)}$

$A^{(k)} = R^{(k)} Q^{(k)}$

eigenvalues are in $\mathrm{diag}(A^{(k)})$.

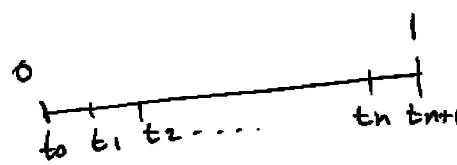
Fundamental differences:

- power method converges to largest eigenvalue (in magnitude)
- QR gives all eigenvalues of matrix.

	Power Method	QR Algo
advantages	• does not need matrix A, only how to apply A to a vector. • Cheap for large matrices • quadratic convergence to eigenvalue	• gives all eigenvalues at once • convergence can be accelerated using shifts to reach cubic convergence for one eigenvalue
disadvantages	• only gives largest eigenvalue • can have slow convergence if largest eigenvalue in magnitude is not simple.	• Can be expensive (specially reduction to tridiagonal form). • needs shifts to accelerate convergence. • needs to store whole matrox

Problem 4

$$\begin{cases} y'' = f(t, y, y') \\ y(0) = \alpha \\ y(1) = \beta \end{cases} \qquad \text{for } t \in [0,1]$$



$t_i = ih$, $i = 0, \ldots, n+1$

$h = \frac{1}{n+1}$

(a) The finite differences approx to BVP can be written as a non-linear system of n equations with n unknowns:

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

<u>note</u>: we ommitted y_0 and y_{n+1} because they are already specified by boundary conditions.

The system of NL eq. can be written as:

$$\underline{F}(\underline{y}) = \underline{0} \quad , \quad \text{where } \underline{F}: \mathbb{R}^n \to \mathbb{R}^n$$

$$\underline{y} \to \begin{bmatrix} F_1(\underline{y}) \\ F_2(\underline{y}) \\ \vdots \\ F_n(\underline{y}) \end{bmatrix}$$

$$F_1(\underline{y}) = \frac{y_2 - 2y_1 + \overset{=\alpha}{\cancel{y_0}}}{h^2} - f\left(t_1, y_1, \frac{y_2 - \overset{\alpha}{\cancel{y_0}}}{2h}\right)$$

$$F_i(\underline{y}) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) \quad , i = 2, \dots, n-1$$

$$F_n(\underline{y}) = \frac{\overset{\beta}{\cancel{y_{n+1}}} - 2y_n + y_{n-1}}{h^2} - f\left(t_n, y_n, \frac{\overset{\beta}{\cancel{y_{n+1}}} - y_{n-1}}{2h}\right).$$

(b) We can write Jacobian $D\underline{F}[\underline{y}]$ in the form:

$$D\underline{F}[\underline{y}] = A - G_1 - G_2$$

where:

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}$$

$$G_1 = \operatorname{diag}\left(f_y\left(t_1, y_1, \frac{y_2 - \alpha}{2h}\right), \dots, f_y\left(t_n, y_n, \frac{\beta - y_{n-1}}{2h}\right)\right)$$

$$G_2 = \frac{1}{2h}\begin{bmatrix} 0 & f_{y'}^{(1)} & & & \\ -f_{y'}^{(2)} & 0 & f_{y'}^{(2)} & & \\ & -f_{y'}^{(3)} & 0 & f_{y'}^{(3)} & \\ & & & \ddots & \\ & & & -f_{y'}^{(n)} & 0 \end{bmatrix}, \quad \text{where: } f_{y'}^{(i)} = f_{y'}\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$$

(c) Since $D\underline{F}[\underline{y}]$ is a tridiagonal matrix, one step of Newton's method for solving $\underline{F}(\underline{y}) = \underline{0}$ <u>costs the same</u> as solving a linear BVP on the same grid.

(d) Typically one would take for $\underline{y}^{(0)}$ the values of line satisfying B.C. [<u>Note:</u> more detail could be asked for this question]

<u>Problem 5</u>

(a) multiplying both sides of BVP by $v \in V$, and integrating:

$$\int_0^1 -u'' v \, dx = \int_0^1 f \, v \, dx \qquad \forall v \in V$$

IBP $\parallel$

$$\cancel{-u'v\Big|_0^1} + \int_0^1 u'v' \, dx$$

$= 0$ because of boundary conditions.

$$\Rightarrow \quad a(u, v) = (f, v) \quad \text{for all } v \in V.$$

with $a(u, v) = \int_0^1 u'v' \, dx$ = bilinear form

$$(f, v) = \int_0^1 f \, v \, dx.$$

(b) Note: The practice exam problem is not very specific as to what version of finite elements to write, so we use simplest one which is:

- Let ϕ_i, $i=1,\dots,n$, be the functions s.t.

$$\phi_i(x_j) = \delta_{ij}, \quad j=1,\dots,n.$$

$$\phi_i\big|_{[x_{j-1},x_j]} \in P_1, \quad \text{for } j=1,\dots n+1,$$

- Compute entries of stiffness matrix:

$$A_{ij} = a(\phi_i, \phi_j) = \int_0^1 \phi_i' \phi_j' \, dx$$

- Compute entries of right hand side.

$$F_i = (f, \phi_i) = \int_0^1 f \phi_i \, dx$$

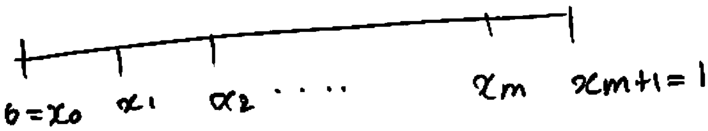
- Solve $AU = F$

- Solution is:

$$u_h(x) = \sum_{i=1}^{n} U_i \phi_i(x)$$

Problem 6

(a) The method of lines simply means we discretize PDE first in space:



$$U_1'(t) = -\frac{a}{2h}\left(U_2(t) - \underbrace{U_0(t)}_{=\,U_{m+1}(t) \text{ by periodic B.C.}}\right)$$

$$U_i'(t) = -\frac{a}{2h}\left(U_{i+1}(t) - U_{i-1}(t)\right), \quad i = 2, \dots, m$$

$$U_{m+1}'(t) = -\frac{a}{2h}\left(\underbrace{U_{m+2}(t)}_{=\,U_1(t) \text{ by periodic B.C.}} - U_m(t)\right).$$

Method of Lines gives system of ODEs.

$$U'(t) = A\,U(t)$$

corner elements because we have periodic B.C.

where
$$A = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & \ddots & \ddots & 1 \\ 1 & & & -1 & 0 \end{bmatrix}$$

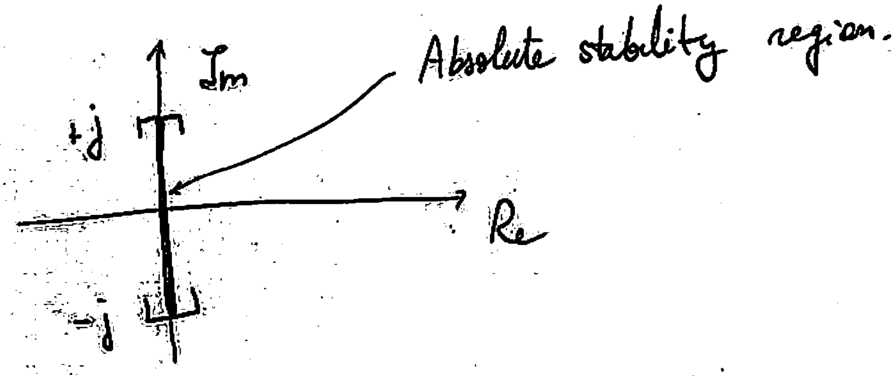
and
$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_{m+1}(t) \end{bmatrix}$$

(b) Discretization in time gives

$$\frac{U^{n+1}-U^{n-1}}{2k} = AU^n$$

$$U^{n+1} = U^{n-1} + 2k\,AU^n$$

(c)



For stability of system of ODEs we need to find k s.t.

$$k\lambda_p(A) \in \text{absolute stability region},\ p = 1, \dots, m+1.$$

Since $\left|\mathrm{Im}\,\lambda_p(A)\right| \le \frac{|a|}{h}$ absolute value important

in order to have:

$$\left|\mathrm{Im}\,k\lambda_p(A)\right| \le 1 \quad \text{we need}$$

$$\frac{k|a|}{h} \le 1 \iff \boxed{k < \frac{h}{|a|}}$$ CFL condition