MATH 5620 – NUMERICAL ANALYSIS II
PRACTICE FINAL EXAM (SPRING 2011)

Problem 1. Consider the multistep method:

$$y_n - 2y_{n-1} + y_{n-2} = h(f_n - f_{n-1}).$$

(a) Is this method implicit or explicit?
(b) Is this method convergent, stable and/or consistent? Justify your answer.

Problem 2. Let A be a $n \times n$ matrix and $Q = \text{diag}(a_{11}, \dots, a_{nn})$.
(a) Compute $Q^{-1}A$ and $I - Q^{-1}A$.
(b) Show that

$$\|I - Q^{-1}A\|_\infty = \max_{1 \le i \le n} \sum_{j=1, j \ne i}^{n} \frac{|a_{ij}|}{|a_{ii}|}.$$

(recall the induced matrix norm $\|B\|_\infty$ is the largest $1-$norm of the rows of B)
(c) Recall the Jacobi method for solving the linear system $Ax = b$,

$$Qx^{(k+1)} = (Q - A)x^{(k)} + b,$$

with Q given as above. Based on part (b), give a condition on relative size of the diagonal and off-diagonal elements of A that guarantees convergence of Jacobi method.

Problem 3.
(a) Write pseudocode for the Power Method and the (unshifted) QR algorithms for symmetric matrices. Please do not worry about stopping criteria. You may assume the availability of a routine for reduction to tridiagonal form.
(b) Explain in your own words the differences, advantages and disadvantages of these two methods.

Problem 4. Assume the non-linear BVP

$$\begin{cases} y'' = f(t, y, y'), \text{ for } t \in [0,1], \\ y(0) = \alpha \\ y(1) = \beta, \end{cases}$$

has a unique solution. Here $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function. Let $t_i = ih$, $i = 0, \dots, n+1$, $h = 1/(n+1)$. Recall the finite difference approximations

$$y'(t_i) = \frac{1}{2h}(y(t_{i+1}) - y(t_{i-1})) + \mathcal{O}(h^2)$$
$$y''(t_i) = \frac{1}{h^2}(y(t_{i+1}) - 2y(t_i) + y(t_{i-1})) + \mathcal{O}(h^2).$$

(a) Let $y_i \approx y(t_i)$. Use the finite difference method to write the approximation to the non-linear BVP in the form

$$\mathbf{F}(\mathbf{y}) = \mathbf{0},$$

with $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{F}(\mathbf{y}) = (F_1(\mathbf{y}), \dots, F_n(\mathbf{y}))^T$ and $\mathbf{y} = (y_1, \dots, y_n)^T$. Please be particularly careful with the equations corresponding to the end of the interval.
(b) Recall Newton's method for solving $\mathbf{F}(\mathbf{y}) = \mathbf{0}$:

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Let y^(0) be an initial guess
for k = 1, ...
  y^(k+1) = y^(k) - (DF[y^(k)])^(-1) F(y^(k))
end
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Find the Jacobian $D\mathbf{F}[\mathbf{y}^{(k)}] = (\nabla F_1(\mathbf{y}^{(k)}), \ldots, \nabla F_n(\mathbf{y}^{(k)}))^T \in \mathbb{R}^{n\times n}$ of $\mathbf{F}$.

(c) How does the computational cost of one Newton's method step compares to solving a linear BVP on the same grid?

(d) What can be used as initial guess $\mathbf{y}^{(0)}$?

Problem 5. Consider the BVP

$$\begin{cases} -u'' = f(x),\ 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases}$$

(a) Let $V = \{v \mid \|v\|_{L^2}^2 + \|v'\|_{L^2}^2 < \infty \text{ and } v(0) = v(1) = 0\}$. Derive the weak formulation of the problem, that is

$$\text{Find } u \in V \text{ such that } a(u,v) = (f,v) \text{ for all } v \in V,$$

explicitly writing $a(u,v)$ and (f,v).

(b) Write pseudocode for solving the problem with piecewise linear (P_1) elements.

Problem 6. Consider the following advection equation (a hyperbolic problem) with periodic boundary conditions,

$$\begin{cases} u_t + au_x = 0,\ t > 0,\ 0 < x < 1, \\ u(0,t) = u(1,t),\ t > 0, \\ u(x,0) = \eta(x),\ 0 < x < 1. \end{cases}$$

(a) The interval $[0,1]$ is discretized with the points $x_i = ih$, $i = 0, \ldots, m+1$, where $h = 1/(m+1) = \Delta x$. Let $U_i(t) \approx u(x_i, t)$. Use the method of lines to approximate the PDE by a system of ODEs $U'(t) = AU(t)$, where $U = [U_1, U_2, \ldots, U_{m+1}]^T$ and A comes from the usual centered differences approximation to u_x.

(b) Discretize in time using the Midpoint method. For a one dimensional IVP $u' = f(u,t)$, the Midpoint method is

$$\frac{u_{n+1} - u_{n-1}}{2k} = f(u_n, t_n).$$

Please use the notation $U_i^n \approx u(x_i, t_n)$ (or $U^n = [U_1^n, U_2^n, \ldots, U_m^n]^T$ in vector form), where $t_n = nk$ and $k = \Delta t$ is the time step.

(c) The absolute stability region for the Midpoint method is the segment between $-j$ and j in the complex plane (here $j = \sqrt{-1}$). Recall that eigenvalues of A are:

$$\lambda_p(A) = -\frac{aj}{h}\sin(2\pi p h),\ p = 1, \ldots, m+1.$$

For a given h, find a condition on k for the system of ODEs to be stable.