PRACTICE MIDTERM SOLUTIONS

Problem 1

$$\|x\| = \|A^{-1}Ax\| \leq \|A^{-1}\|\,\|Ax\|$$

$$\Rightarrow \quad \|Ax\| \geq \|A^{-1}\|^{-1}\,\|x\|.$$

Problem 2

$$y = x + t^* v$$

$$t^* = \frac{v^T(b - Ax)}{v^T A v}$$

$$v^T(b - Ay) = v^T(b - Ax - t^* A v)$$

$$= v^T(b - Ax) - t^* v^T A v$$

$$= v^T(b - Ax) - \frac{v^T(b - Ax)}{\cancel{v^T A v}}\,\cancel{v^T A v}$$

$$= 0$$

Problem 3

(a) $$u(t) = \frac{\partial y(t;z)}{\partial z}$$

$$u'' = \frac{\partial y''(t;z)}{\partial z} = \frac{\partial}{\partial z}\left[f(t, y(t;z), y'(t;z))\right]$$

$$= \frac{\partial t}{\partial z} f_t(t,y,y') + \frac{\partial y}{\partial z} f_y(t,y,y') + \frac{\partial y'}{\partial z} f_{y'}(t,y,y')$$

$$= u f_y(t,y,y') + u' f_{y'}(t,y,y')$$

$$u(\alpha) = \frac{\partial y(a;z)}{\partial z} = 1$$

$$u'(a) = \frac{\partial y'(a;z)}{\partial z} = 0$$

(b) $$\phi(z) = y(b;z) - \beta$$

$$\phi'(z) = \frac{\partial y(b;z)}{\partial z} = u(b)$$

(c) Let $y_1 = y$ and $y_2 = u$ then:

$$(\text{Sys}) \quad \begin{cases} y_1' = y_3 \\ y_2' = y_4 \\ y_3' = f(t, y_1, y_3) \\ y_4 = y_2 f(t, y_1, y_3) + y_4 f(t, y_1, y_3) \end{cases}$$

I.C.: $y_1(a) = y(a) = z$ | $y_3(a) = y'(a) = \alpha$

$y_2(a) = u(a) = 1$ | $y_4(a) = u'(a) = 0$

(d) For initial condition we take equation of line satisfying B.C. of BVP:

$$y(t) = \alpha(t-b) + \beta$$

$$z_0 = y(a) = \alpha(a-b) + \beta$$

for $k = 0, 1, 2, \dots$

solve (SYS) with I.C. $(z_k, 1, \alpha, 0)$

$$z_{k+1} = z_k - \frac{y_1(b) - \beta}{y_2(b)}$$

IF $|z_{k+1} - z_k| < TOL$ STOP

another possible stopping criterion is:

$$|y_1(b) - \beta| < TOL.$$

Problem 4

(a) $$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2!} y''(t_i) + \frac{h^3}{3!} y^{(3)}(t_i) + \frac{h^4}{4!} y^{(4)}(\xi_i^+)$$

$$y(t_{i-1}) = y(t_i) - h y'(t_i) + \frac{h^2}{2!} y''(t_i) - \frac{h^3}{3!} y^{(3)}(t_i) + \frac{h^4}{4!} y^{(4)}(\xi_i^-)$$

where $\xi_i^+ \in [t_i, t_{i+1}]$ and $\xi_i^- \in [t_{i-1}, t_i]$

Adding both equations and solving for $y''(t_i)$ we get:

$$y''(t_i) = \frac{y(t_{i+1}) - 2y(t_i) + y(t_{i-1})}{h^2} - \frac{h^2}{24} \underbrace{\left[y^{(4)}(\xi_i^+) + y^{(4)}(\xi_i^-) \right]}_{= 2 y^{(4)}(\xi_i)}$$

for some $\xi_i \in [t_{i-1}, t_{i+1}]$

(b) and (c):

$$L = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ & & & 1 & -2 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$$D = \frac{1}{2h}\begin{bmatrix} 0 & p_1 & & & \\ -p_2 & 0 & p_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & p_{n-1} \\ & & & p_n & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$$Q = \operatorname{diag}(q_1, q_2, \dots, q_n)$$

where we use notation:

$$p_i \equiv p(t_i)$$
$$q_i \equiv q(t_i)$$
$$r_i \equiv r(t_i)$$

FD approx is:

$$(\underbrace{L - D - Q}_{A})Y = B, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$B = \begin{bmatrix} r_1 - \alpha/h^2 - p_1/2h \\ r_2 \\ \vdots \\ r_{n-1} \\ r_n - \beta/h^2 + p_n/2h \end{bmatrix}$$

Problem 5

(a) Here:

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix} \in \mathbb{R}^{m \times m}$$

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_m(t) \end{bmatrix}$$

$$V = \begin{bmatrix} \eta(x_1) \\ \eta(x_2) \\ \vdots \\ \eta(x_m) \end{bmatrix}$$

(b)

$$U^{n+1} = U^n + kAU^n$$
$$= (I + kA)U^n$$

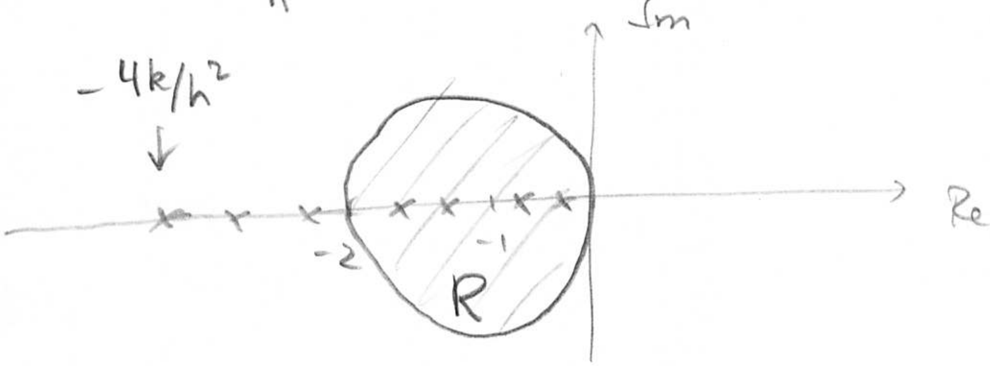
(c)

$$U^1 = (I+kA)V$$
$$U^2 = (I+kA)^2 V$$
$$\vdots$$
$$U^n = (I+kA)^n V$$

(d) we need $k\lambda_p(A) \in$ absolute stability region for all p.

$$\lambda_p(A) = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right)$$

$$\Rightarrow \quad -\frac{4k}{h^2} \leq k\lambda_p(A) \leq 0$$



For $k\lambda_p(A) \in R$ we need:

$$-2 \leq \frac{-4k}{h^2}$$

$$\Rightarrow \quad \boxed{\frac{k}{h^2} \leq \frac{1}{2}}$$