MATH 5620 – NUMERICAL ANALYSIS II
PRACTICE MIDTERM EXAM

Note: This exam is longer than the actual midterm (maybe 60-75min).

Problem 1. Show that if A is invertible
$$\|Ax\| \geq \|x\|\|A^{-1}\|^{-1}.$$

Problem 2. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and $v \in \mathbb{R}^n$ be a non-zero vector. Consider
$$y = x + t^* v,$$
where
$$t^* = \frac{v^T(b - Ax)}{v^T A v}.$$
Show that
$$v^T(b - Ay) = 0.$$

Problem 3. The goal of this problem is to design a non-linear shooting method for solving a non-linear BVP with mixed type boundary conditions:
$$\begin{cases} y'' = f(t, y, y'), \text{ for } t \in [a, b], \\ y'(a) = \alpha, \\ y(b) = \beta. \end{cases} \tag{1}$$
Let $y = y(t; z)$ be the solution to the IVP
$$\begin{cases} y'' = f(t, y, y'), \text{ for } t \in [a, b], \\ y(a) = z, \\ y'(a) = \alpha. \end{cases} \tag{2}$$
and let $\phi(z) = y(b; z) - \beta$.

(a) Let $u(t) = \frac{\partial y(t;z)}{\partial z}$. By differentiating (2) with respect to z, show that u solves
$$\begin{cases} u'' = u f_y(t, y, y') + u' f_{y'}(t, y, y'), \text{ for } t \in [a, b], \\ u(a) = 1 \\ u'(a) = 0. \end{cases} \tag{3}$$
(b) Show that $\phi'(z) = u(b)$.

(c) Write (2) and (3) as a system of four first-order equations (do not forget initial conditions).

(d) Assuming the availability of a routine for solving systems of first order equations, write pseudocode for solving (1), based on Newton's method for finding z such that $\phi(z) = 0$.

Problem 4. Consider the linear BVP
$$\begin{cases} y'' = py' + qy + r \\ y(0) = \alpha, \ y(1) = \beta \end{cases}$$
where p, q and r are smooth functions defined on $[a, b]$. Let $t_i = ih$, where $i = 0, \ldots, n+1$ and $h = 1/(n + 1)$.

(a) Use the Taylor expansions of $y(t_{i+1})$ and $y(t_{i-1})$ around $t = t_i$ to show that
$$y''(t_i) = \frac{1}{h^2}(y(t_{i+1}) - 2y(t_i) + y(t_{i-1})) + \mathcal{O}(h^2).$$

(b) Recall the centered differences approximation

$$y'(t_i) = \frac{1}{2h}(y(t_{i+1}) - y(t_{i-1})) + \mathcal{O}(h^2).$$

Write the finite difference approximation to the problem, using the notation $y_i \approx y(t_i)$. Since the boundary conditions are $y_0 = \alpha$ and $y_{n+1} = \beta$, there are only n equations.

(c) Write the finite differences approximation as a system $AY = B$ with n unknowns.

Problem 5. Consider the following parabolic problem (e.g. heat equation):

$$\begin{cases} u_t = u_{xx},\ t > 0,\ 0 < x < 1, \\ u(0,t) = u(1,t) = 0,\ t > 0, \\ u(x,0) = \eta(x),\ 0 < x < 1. \end{cases}$$

(a) The interval $[0,1]$ is discretized with the points $x_i = ih$, $i = 0, \dots, m+1$, where $h = 1/(m+1) = \Delta x$. Let $U_i(t) \approx u(x_i, t)$. Use the method of lines to approximate the PDE by a system of ODEs

$$\begin{cases} U'(t) = AU(t),\ t > 0 \\ U(0) = V, \end{cases} \tag{SYS}$$

where $U = [U_1, U_2, \dots, U_m]^T$ and A comes from the usual three point stencil finite differences approximation to u_{xx}.

(b) Discretize (SYS) in time using Euler's method. Please use the notation $U_i^n \approx u(x_i, t_n)$ (or $U^n = [U_1^n, U_2^n, \dots, U_m^n]^T$ in vector form), where $t_n = nk$ and $k \equiv \Delta t$ is the time step.

(c) Show that the iterates in Euler's method satisfy

$$U^n = (I + kA)^n V.$$

(d) The absolute stability region for Euler's method is the disk $\{z \mid |z+1| \leq 1\}$ in the complex plane. Recall that eigenvalues of A are:

$$\lambda_p(A) = \frac{2}{h^2}(\cos(p\pi h) - 1),\ p = 1, \dots, m.$$

For a given h, find a condition on k for the stability of Euler's method applied to (SYS).