Problem 1. Show that if $A$ is invertible
\[ \|Ax\| \geq \|x\| \|A^{-1}\|^{-1}. \]

Problem 2. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix and $v \in \mathbb{R}^n$ be a non-zero vector. Consider
\[ y = x + t^*v, \]
where
\[ t^* = \frac{v^T(b - Ax)}{v^TAv}. \]
Show that
\[ v^T(b - Ay) = 0. \]

Problem 3. The goal of this problem is to design a non-linear shooting method for solving a non-linear BVP with mixed type boundary conditions:
\[
\begin{cases}
  y'' = f(t, y, y'), & t \in [a, b], \\
  y'(a) = \alpha, \\
  y(b) = \beta.
\end{cases}
\]
(1)
Let $y = y(t; z)$ be the solution to the IVP
\[
\begin{cases}
  y'' = f(t, y, y'), & t \in [a, b], \\
  y(a) = z, \\
  y'(a) = \alpha.
\end{cases}
\]
(2)
and let $\phi(z) = y(b; z) - \beta$.
(a) Let $u(t) = \frac{\partial y(t; z)}{\partial z}$. By differentiating (2) with respect to $z$, show that $u$ solves
\[
\begin{cases}
  u'' = uf_y(t, y, y') + u'f_y'(t, y, y'), & t \in [a, b], \\
  u(a) = 1 \\
  u'(a) = 0.
\end{cases}
\]
(3)
(b) Show that $\phi'(z) = u(b)$.
(c) Write (2) and (3) as a system of four first-order equations (do not forget initial conditions).
(d) Assuming the availability of a routine for solving systems of first order equations, write pseudocode for solving (1), based on Newton’s method for finding $z$ such that $\phi(z) = 0$.

Problem 4. Consider the linear BVP
\[
\begin{cases}
  y'' = py' + qy + r \\
  y(0) = \alpha, & y(1) = \beta
\end{cases}
\]
where $p$, $q$ and $r$ are smooth functions defined on $[a, b]$. Let $t_i = ih$, where $i = 0, \ldots, n + 1$ and $h = 1/(n + 1)$.
(a) Use the Taylor expansions of $y(t_{i+1})$ and $y(t_{i-1})$ around $t = t_i$ to show that
\[ y''(t_i) = \frac{1}{h^2}(y(t_{i+1}) - 2y(t_i) + y(t_{i-1})) + O(h^2). \]
(b) Recall the centered differences approximation
\[
y'(t_i) = \frac{1}{2h}(y(t_{i+1}) - y(t_{i-1})) + O(h^2).
\]
Write the finite difference approximation to the problem, using the notation \( y_i \approx y(t_i) \). Since the boundary conditions are \( y_0 = \alpha \) and \( y_{n+1} = \beta \), there are only \( n \) equations.

(c) Write the finite differences approximation as a system \( AY = B \) with \( n \) unknowns.

**Problem 5.** Consider the following parabolic problem (e.g. heat equation):
\[
\begin{aligned}
&u_t = u_{xx}, \quad t > 0, \quad 0 < x < 1, \\
&u(0, t) = u(1, t) = 0, \quad t > 0, \\
&u(x, 0) = \eta(x), \quad 0 < x < 1.
\end{aligned}
\]

(a) The interval \([0, 1]\) is discretized with the points \( x_i = ih, \ i = 0, \ldots, m + 1 \), where \( h = 1/(m + 1) = \Delta x \). Let \( U_i(t) \approx u(x_i, t) \). Use the method of lines to approximate the PDE by a system of ODEs
\[
\begin{aligned}
&U'(t) = AU(t), \quad t > 0 \\
&U(0) = V,
\end{aligned}
\]
where \( U = [U_1, U_2, \ldots, U_m]^T \) and \( A \) comes from the usual three point stencil finite differences approximation to \( u_{xx} \).

(b) Discretize (SYS) in time using Euler’s method. Please use the notation \( U^n_i \approx u(x_i, t_n) \) (or \( U^n = [U^n_1, U^n_2, \ldots, U^n_m]^T \) in vector form), where \( t_n = nk \) and \( k \equiv \Delta t \) is the time step.

(c) Show that the iterates in Euler’s method satisfy
\[
U^n = (I + kA)^n V.
\]

(d) The absolute stability region for Euler’s method is the disk \( \{ z | |z + 1| \leq 1 \} \) in the complex plane. Recall that eigenvalues of \( A \) are:
\[
\lambda_p(A) = \frac{2}{h^2}(\cos(p\pi h) - 1), \quad p = 1, \ldots, m.
\]
For a given \( h \), find a condition on \( k \) for the stability of Euler’s method applied to (SYS).