MATH 5620 NUMERICAL ANALYSIS II
HOMEWORK 4, DUE FRIDAY MARCH 19 2010

Problem 1 Consider the Poisson's equation

$$\begin{aligned} \Delta u &= f(x,y) \quad \text{for } x \in [0,1] \text{ and } y \in [0,1] \\ u(x,y) &= 0 \quad \text{if } x = 0 \text{ or } x = 1 \text{ or } y = 0 \text{ or } y = 1. \end{aligned}$$

With

$$f(x,y) = \sin(\pi x)\sin(2\pi y),$$

the true solution is

$$u(x,y) = -f(x,y)/(5\pi^2).$$

Use the finite difference method with $x_i = ih$, $i = 0, \ldots, n+1$ and $y_j = jh$, $j = 0, \ldots, n+1$, for the values $n = 10, 50, 100$ and $h = 1/(n+1)$. Compute the maximum absolute error in your approximation and produce a log-log plot with h in the abscissa and the error in the ordinate. Is this plot consistent with the expected $\mathcal{O}(h^2)$ convergence rate?

Notes:

- You may find it easier to write the discretization matrix with Matlab's `kron` (in Octave replace by `spkron`). See class notes: `http://www.math.utah.edu/~fguevara/math5620_s10/na006.pdf`
- You may use Matlab's backslash to solve the system.
- Your system matrix should be $n^2 \times n^2$. Think of using matrix operations to put values in lexicographic ordering:

```
x = linspace(0,1,n+2); y = linspace(0,1,n+2);
[X,Y] = meshgrid(x,y);
u = @(x,y) .... % some function
U = u(X,Y);
```

  Then the matrix U is such that `U(i,j) = u(x(j),y(i))`. Since the vector `U(:)` contains the columns of `U` concatenated, it corresponds to ordering the nodes by x and then by y as in the following 3×3 example:

$$\begin{array}{cccc} & 3 & 6 & 9 \\ y\uparrow & 2 & 5 & 8 \\ & 1 & 4 & 7 \\ & \rightarrow & & \\ & x & & \end{array}$$

  where the arrows indicate the direction of increasing values of the corresponding variables.

Problem 2 Consider the parabolic PDE (heat equation)

$$
\begin{aligned}
u_t &= u_{xx} && \text{for } t > 0 \text{ and } x \in [0,1],\\
u(x,0) &= \eta(x) && \text{for } x \in [0,1],\\
u(0,t) = u(1,t) &= 0 && \text{for } t > 0,
\end{aligned}
$$

Use the Crank-Nicholson method with the space discretization $x_i = ih$, $i = 0, \ldots, n+1$, $h = 1/(n+1)$, $n = 100$ and time discretization $k = 1/1000$ to approximate the solution for the initial conditions

(a) $\eta(x) = \sin(\pi x)$

(b) $\eta(x) = \sin(\pi x) + \sin(10\pi x)$

Please include snapshots of both solutions at times $t = 2k$ and $t = 5k$.

Notes:

- With these particular boundary conditions the method can be written as

  $$U^{n+1} = (I - (k/2)A)^{-1}(I + (k/2)A)U^n$$

  where A is the usual finite difference discretization of the 1D Laplacian.
- You may use Matlab's backslash to solve the systems at each iteration.