

## MATH 5620 NUMERICAL ANALYSIS II

### MIDTERM REVIEW SHEET

#### CHAPTER 5 – INITIAL VALUE PROBLEMS

- §5.1 Lipschitz condition, well-posedness, existence and uniqueness theorem for first order IVP.
- §5.2-5.3: Taylor series method, know how to derive them. Notion of order, local truncation error and global truncation error (in general). How to find local truncation error for Taylor series method.
- §5.4: Runge-Kutta methods. Advantages over Taylor series methods. Know general form of second order Runge-Kutta method, and why the parameters are related (match Taylor series). You don't need to know all the RK2 methods by their names. Why are RK methods of order higher than 4 less attractive than RK4?
- §5.5: Runge-Kutta-Fehlberg method. You have to know the basic idea (using embedded methods). Please do not learn the method by heart or its coefficients. Need to understand the derivation of update for  $h$ .
- §5.6 Multistep methods. Understand the relationship with integration formulas. Use the undetermined coefficient method. Difference between Adams-Bashforth and Adams-Moulton. What is a predictor-corrector method? How is it related to a fixed point iteration?
- Analysis of linear multi-step methods. Know the linear functional  $L$  and how it can be used to get the order or the local truncation error of a method. You do not need to know the formulas for the  $d_j$  exactly, but you should know where they come from.
- §5.7 Variable step-size multistep methods. You need to know the big idea to obtain adaptivity. How does this method compare to Runge-Kutta-Fehlberg from the practical point of view? As for RKF you may be asked to estimate the size  $h$  to get a desired precision.
- §5.8 Extrapolation methods: Know the main idea (clever linear combinations to get error cancellations) and how to apply it to an example.
- §5.9 Higher order equations and systems. Know how to transform an algorithm for the scalar case to system case. Know how to transform a higher order equation into a system.
- §5.10 Stability, only for *linear* multistep methods. Difference between implicit/explicit. Notion of convergent method. Consistency and stability notions and how they relate to convergence. Know how

to determine if a given linear multistep method is consistent, stable or convergent. (you don't need to know the proof that convergence implies stability and consistency).

- Difference equations. Know how to solve a difference equation when roots of characteristic polynomial.
- §5.11 Stiff differential equations. A-stability and how to determine if a linear multi-step method is A-stable. What is the region of absolute stability and what does it mean? (both are linked to how the method fares with the simple problem  $y' = \lambda y; y(0) = 1$  with a given time step  $h$ ).

## CHAPTER 11 – BOUNDARY VALUE PROBLEMS

- Theorem giving sufficient condition for the problem to have a unique solution.
- §11.1–11.2: Linear shooting method and Non-Linear Shooting method. You should know the basic ideas behind these two methods. Without knowing the formulas by heart you should be able to linearly combine two solutions to a linear IVP to get the solution to a linear BVP. Also you should understand how non-linear shooting works (with Newton's method)
- §11.3–11.4: Finite Difference Methods. For linear BVP know how to construct the linear system. For Non-Linear BVP: know how to setup the non-linear system (Jacobian etc. . . ) and Newton's method.

## CHAPTER 12 – NUMERICAL SOLUTION TO PARTIAL DIFFERENTIAL EQUATIONS

**Note:** What we saw in class is significantly different from what is covered in the class textbook, so please refer to the class notes.

- **Elliptic problems:** (Laplace equation  $\Delta u = f$ ) Five point stencil for Laplacian and how the discretization works. Know that the local truncation error is  $\mathcal{O}(h^2)$ .
- **Parabolic problems:** (heat equation  $u_t = \kappa \Delta u$ ).
  - Method of lines discretization and how relationship between Trapezoidal rule and Crank-Nicholson.
  - Please do not remember the expressions for the eigenvalues of the discrete Laplacian. However you should be able to use these expressions to determine the region of stability of a discretization method: for example for Euler's method you should be able to get that  $k/h^2 < 1/2$  from the expression of the eigenvalues for the discrete Laplacian.
  - Know the idea of LOD and ADI for multidimensional problems. What does one gain by using these “operator splitting” methods?
- **Hyperbolic problems**

- You need to understand Leapfrog, upwind, Lax-Friedrich and Lax-Wendroff method. It is not necessary to remember these methods and/or their derivations by heart.
- For a given method you should be able to show where the eigenvalues of the iteration matrix lie (given the eigenvalues of the discrete Laplacian and those of the matrix representing the difference operator) and deduce the CFL condition from the stability conditions (i.e. the eigenvalues of the iteration matrix should be inside the absolute stability region of the method used for the time discretization).
- **Finite Elements** (only for Laplace equation  $-\Delta u = f$ )
  - Know weak formulation (1D and 2D)
  - Ritz-Galerkin approximation, Galerkin orthogonality, basic error estimate (with the energy norm), optimality of the Ritz-Galerkin approximation in the energy norm.
  - Know what the Sobolev norms  $\|\cdot\|_{L^2(\Omega)}$  and  $\|\cdot\|_{H^k(\Omega)}$  as well as the semi-norm  $|\cdot|_{H^k(\Omega)}$  mean. There will be no derivation of error estimates (and you do not need to know any other error estimate than the basic one) but you may be asked to piece together inequalities to obtain an error estimate.
  - Know what are the local and global interpolants.
  - Know how to assemble the stiffness matrix and right hand side for 1D P1 elements (2 chain rules + change of integration variable).

## CHAPTER 9 – EIGENVALUES

**Note:** What we saw in class is significantly different from what is covered in the class textbook, so please refer to the class notes.

- Know how to write pseudocode for the power method, symmetric power method, inverse iteration and Rayleigh quotient iteration.
- You should be able to say what are the advantages/disadvantages of such methods from the computational point of view.
- Know the idea behind Householder reflectors and how they can be used to reduce a matrix to Hessenberg (or tridiagonal) form. You do not need to know how to choose the Householder reflector that is more numerically stable.
- You should know the basic version of the QR algorithm and the simultaneous iteration algorithm (which can be used to study the QR algorithm see p71 in `na011.pdf`). What is the computational cost of one iteration of the QR algorithm? What does it compute? Where are the eigenvalues and eigenvectors when the algorithm terminates?

## OTHER CONSIDERATIONS

- Understanding of the methods will be favored over simple memorization.
- You should be able to evaluate a method based on its practical merits (does it take too many function evaluations?...)
- Expect to write pseudo-code
- You are expected to know how to solve  $2 \times 2$  systems (and simple  $3 \times 3$  or triangular systems). Also you should know how to find the roots of a quadratic or a simple cubic or quartic polynomial.
- A calculator is not needed for the exam, and is not allowed.
- Two one-sided (or equivalent) letter sized, handwritten cheat sheet is allowed.
- No books or other notes are allowed.