MATH 5620/6865

PRACTICE FINAL SOLUTIONS

Problem 1 $$y_n - 2y_{n-1} + y_{n-2} = h\,(f_n - f_{n-1})$$

(a) This is an implicit method because y_n appears on RHS.

(b) The polynomials associated with the method are:

$$p(z) = z^2 - 2z + 1 \qquad q(z) = z^2 - z$$

$$p'(z) = 2z - 2$$

$$\left.\begin{aligned} p(1) &= 1 - 2 + 1 = 0 \\ p'(1) &= 0 = q(1) \end{aligned}\right\} \Rightarrow \text{method is consistent}$$

To check stability we need the roots of $p(z)$:

$$p(z) = (z-1)^2 \Rightarrow \text{root } 1 \text{ has multiplicity } 2.$$

$\Rightarrow$ method is not stable because there are roots with $|z| = 1$ and multiplicity ≥ 1.

$\Rightarrow$ method is not convergent.

Problem 2

A is assumed to be symmetric ($A = A^T$)

(a)

Power method

$v^{(0)}$ = some vector with $\|v^{(0)}\| = 1$

for $k = 1, 2, \dots$

$$w^{(k)} = A v^{(k-1)}$$
$$v^{(k)} = w^{(k)} / \|w^{(k)}\|$$
$$\lambda^{(k)} = v^{(k)T} A v^{(k)}$$

QR algorithm

$A^{(0)} = Q_0^* A Q_0 =$

(reduction to tridiagonal form)

for $k = 1, 2, \dots$

$$Q^{(k)} R^{(k)} = A^{(k-1)}$$
$$A^{(k)} = R^{(k)} Q^{(k)}$$

eigenvalues are in diagonal of $A^{(k)}$.

Fundamental differences:

- power method converges to largest eigenvalue (in magnitude)
- QR gives all eigenvalues of matrix.

	Power method	QR algo
advantages	• does not need matrix A, only how to apply A to a vector • cheap for large matrices • quadratic convergence to eigenvalue	• gives all eigenvalues at once • convergence can be accelerated using shifts to reach cubic convergence for one eigenvalue.
disadvantages	• only gives largest eigenvalue • can have slow convergence if largest eigenvalue in magnitude is not simple.	• can be expensive (specially reduction to tridiagonal form) • needs shifts to accelerate convergence. • needs to store whole matrix

Problem 3

(a) The method of lines simply means we discretize PDE first in space :

$$0 = x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_m \quad x_{m+1}$$

$$U_1'(t) = -\frac{a}{2h}\left(U_2(t) - \overbrace{U_0(t)}^{=\,U_{m+1}(t) \text{ by periodic B.C.}}\right)$$

$$\vdots$$

$$U_i'(t) = -\frac{a}{2h}\left(U_{i+1}(t) - U_{i-1}(t)\right) \qquad i = 2 .. m$$

$$U_{m+1}'(t) = -\frac{a}{2h}\left(\underbrace{U_{m+2}(t)}_{=\,U_1(t) \text{ by periodic B.C.}} - U_m(t)\right)$$

Method of lines gives:

$$U'(t) = AU(t)$$

corner elements are important for accurately describing periodic B.C.

where

$$A = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & \ddots & \ddots & 1 \\ 1 & & & -1 & 0 \end{bmatrix}$$

and

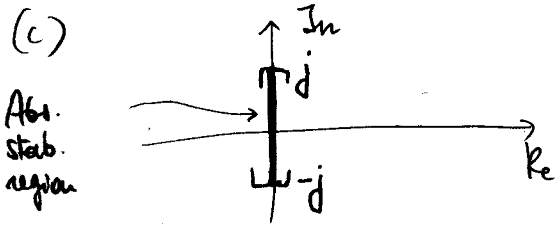
$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_{m+1}(t) \end{bmatrix}$$

(b) Discretization in time gives:

$$\frac{U^{n+1} - U^{n-1}}{2k} = AU^n$$

$$U^{n+1} = U^{n-1} + 2kAU^n$$

(c)



For stability of system of ODE's we need to find k s.t.

$$k\lambda_p(A) \in \text{absolute stability region}, \quad p = 1, \ldots, m+1.$$

Since

$$|\mathrm{Im}\,\lambda_p(A)| \leq \frac{|a|}{h}$$ ← absolute value is important here

In order to have:

$$|\mathrm{Im}\,k\lambda_p(A)| \leq 1 \text{ we need}$$

$$\frac{k|a|}{h} \leq 1 \iff \boxed{k \leq \frac{h}{|a|}}$$ (~ CFL condition)

Problem 4

(a) Find $u_h \in V_h$ s.t.

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

(b) Galerkin orthogonality:

$$a(u, v_h) = (f, v_h) \qquad \text{(Weak formulation)}$$
$$a(u_h, v_h) = (f, v_h) \qquad \text{(Ritz-Galerkin)}$$

$$a(u - u_h, v_h) = 0 \quad \text{for all } v_h \in V_h$$

To show inequality:

$$\|u - u_h\|_E^2 = a(u - u_h, u - u_h)$$
$$\underset{(a(\cdot,\cdot)\text{ bilin. form})}{=} a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h)$$
$$\underset{(\text{Galerkin }\perp)}{=} a(u - u_h, u - v_h) \quad \text{for all } v_h \in V_h$$
$$\underset{(\text{Cauchy Schwarz})}{\leq} \|u - u_h\|_E \, \|u - v_h\|_E \quad \text{for all } v_h \in V_h$$

$$\Rightarrow \boxed{\|u - u_h\|_E \leq \|u - v_h\|_E \quad \text{for all } v_h \in V_h}$$

(c) Since $I_{2h} u \in V_h$:

$$\|u - u_h\|_E \underset{(I1)}{\leq} \|u - I_{2h} u\|_E \underset{(I2)}{\leq} C \|u - I_{2h} u\|_{H^1(\Omega)}$$
$$\underset{(I3)}{\leq} C h^k |u|_{H^{k+1}(\Omega)}$$

Using (I2) we obtain inequality:

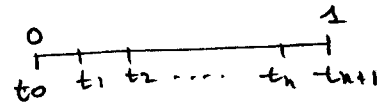
$$\|u - u_h\|_{H^1(\Omega)} \leq C h^k |u|_{H^{k+1}(\Omega)}$$

Now using Cea's lemma (I4) we get the desired result:

$$\|u-u_h\|_{L^2(\Omega)} \underset{(I4)}{\leq} Ch\,\|u-u_h\|_{H^1(\Omega)} \leq Ch^{k+1}\,|u|_{H^{k+1}(\Omega)}$$

Problem 5

$$\begin{cases} y'' = f(t, y, y') & \text{for } t \in [0,1] \\ y(0) = \alpha \\ y(1) = \beta \end{cases}$$



$t_i = ih, \quad i = 0, \dots, n+1$

$h = \frac{1}{n+1}$

(a) The finite difference approx to BVP can be written as a non-linear system of n equations with n unknowns $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

(note we do not include the values y_0, y_{n+1} because they are already specified by boundary conditions)

$$\underline{F}(\underline{y}) = \underline{0} \quad \text{where} \quad \underline{F}: \mathbb{R}^n \to \mathbb{R}^n$$

$$\underline{y} \to \begin{bmatrix} F_1(\underline{y}) \\ F_2(\underline{y}) \\ \vdots \\ F_n(\underline{y}) \end{bmatrix}$$

$$F_1(\underline{y}) = \frac{y_2 - 2y_1 + \overset{\alpha}{\cancel{y_0}}}{h^2} - f\left(t_1, y_1, \frac{y_2 - \overset{\alpha}{\cancel{y_0}}}{2h}\right)$$

$$F_i(\underline{y}) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right), \quad i = 2 \dots n-1$$

$$F_n(\underline{y}) = \frac{\overset{\beta}{\cancel{y_{n+1}}} - 2y_n + y_{n-1}}{h^2} - f\left(t_n, y_n, \frac{\overset{\beta}{\cancel{y_{n+1}}} - y_{n-1}}{2h}\right).$$

(b) We can write Jacobian $D\underline{F}[\underline{y}]$ in the form:

$$D\underline{F}[\underline{y}] = A - G_1 - G_2$$

where:

$$A = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} \in \mathbb{R}^{m \times n},$$

$$G_1 = \operatorname{diag}\left(f_y\left(t_1, y_1, \frac{y_2 - \alpha}{2h}\right), \ldots, f_y\left(t_n, y_n, \frac{\beta - y_{n-1}}{2h}\right)\right).$$

$$G_2 = \frac{1}{2h} \begin{bmatrix} 0 & f_{y'}^{(1)} & & & \\ -f_{y'}^{(2)} & 0 & f_{y'}^{(2)} & & \\ & f_{y'}^{(3)} & 0 & f_{y'}^{(3)} & \\ & & \ddots & & f_{y'}^{(m-1)} \\ & & & -f_{y'}^{(m)} & 0 \end{bmatrix}$$ and $f_{y'}^{(i)} \equiv f_{y'}\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$

(c) Since $D\underline{F}[\underline{y}]$ is a tridiagonal matrix, one step of Newton's method for solving $\underline{F}(\underline{y}) = \underline{0}$ <u>costs the same</u> as solving a linear BVP on the same grid.

(d) Typically one would take for $\underline{y}^{(0)}$ the values of line satisfying B.C.:

$$\underline{y}_i^{(0)} = \alpha + ih(\beta - \alpha), \quad i = 0, \ldots, m+1$$

(check: $y_0^{(0)} = \alpha$ and $y_{m+1}^{(0)} = \beta$)

Problem 6

(a) $$\tau_{i+1}(h) = y_{i+1} - y(t_{i+1}) + \mathcal{O}(h^3)$$
$$\tilde{\tau}_{i+1}(h) = \tilde{y}_{i+1} - y(t_{i+1}) + \mathcal{O}(h^4)$$

Thus:
$$\tau_{i+1}(h) = y_{i+1} - \tilde{y}_{i+1} + \underbrace{\underbrace{\tilde{y}_{i+1} - y(t_{i+1})}_{=\mathcal{O}(h^4)} + \mathcal{O}(h^3)}_{=\mathcal{O}(h^3)}$$
$$= y_{i+1} - \tilde{y}_{i+1} + \mathcal{O}(h^3)$$

(b) If $\tau_{i+1}(h) = K h^2$ we have:
$$\tau_{i+1}(qh) = K q^2 h^2 = q^2 \tau_{i+1}(h)$$

(c) Combining (a) and (b) we have:
$$\tau_{i+1}(qh) \approx q^2 (y_{i+1} - \tilde{y}_{i+1})$$
We would like to find q for which:
$$|\tau_{i+1}(qh)| < \varepsilon$$
$$|q^2 (y_{i+1} - \tilde{y}_{i+1})| < \varepsilon$$
$$q^2 < \frac{\varepsilon}{|y_{i+1} - \tilde{y}_{i+1}|}$$
$$\boxed{q < \left(\frac{\varepsilon}{|y_{i+1} - \tilde{y}_{i+1}|}\right)^{\frac{1}{2}}}$$