MATH 5620/6865 – NUMERICAL ANALYSIS II
PRACTICE FINAL EXAM (SPRING 2010)

Problem 1. Consider the multistep method:

$$y_n - 2y_{n-1} + y_{n-2} = h(f_n - f_{n-1}).$$

(a) Is this method implicit or explicit?
(b) Is this method convergent, stable and/or consistent? Justify your answer.

Problem 2.
(a) Write pseudocode for the Power Method and the (unshifted) QR method for symmetric matrices. Please do not worry about stopping criteria.
(b) Explain in your own words the differences, advantages and disadvantages of these two methods.

Problem 3. Consider the following advection equation (a hyperbolic problem) with periodic boundary conditions,

$$\begin{cases} u_t + au_x = 0,\ t > 0,\ 0 < x < 1, \\ u(0,t) = u(1,t),\ t > 0, \\ u(x,0) = \eta(x),\ 0 < x < 1. \end{cases}$$

(a) The interval $[0,1]$ is discretized with the points $x_i = ih$, $i = 0, \ldots, m+1$, where $h = 1/(m+1) = \Delta x$. Let $U_i(t) \approx u(x_i,t)$. Use the method of lines to approximate the PDE by a system of ODEs $U'(t) = AU(t)$, where $U = [U_1, U_2, \ldots, U_{m+1}]^T$ and A comes from the usual centered differences approximation to u_x.
(b) Discretize in time using the Midpoint method. For a one dimensional IVP $u' = f(u,t)$, the Midpoint method is

$$\frac{u_{n+1} - u_{n-1}}{2k} = f(u_n, t_n).$$

Please use the notation $U_i^n \approx u(x_i, t_n)$ (or $U^n = [U_1^n, U_2^n, \ldots, U_m^n]^T$ in vector form), where $t_n = nk$ and $k = \Delta t$ is the time step.
(c) The absolute stability region for the Midpoint method is the segment between $-j$ and j in the complex plane (here $j = \sqrt{-1}$). Recall that eigenvalues of A are:

$$\lambda_p(A) = -\frac{aj}{h}\sin(2\pi ph),\ p = 1, \ldots, m+1.$$

For a given h, find a condition on k for the system of ODEs to be stable.

Problem 4. Let V be an appropriate subspace of functions defined on a domain Ω and assume the weak formulation of a PDE gives the problem

$$\text{Find } u \in V \text{ such that } a(u, v) = (f, v) \text{ for all } v \in V,$$

here $a(u, v)$ is a bilinear form, (f, v) is an inner product and V is an appropriate subspace of functions.

(a) Let V_h be a finite dimensional subspace of V. Write down the Ritz-Galerkin formulation of the problem.
(b) Let u be the solution to the weak formulation of the PDE and u_h be the solution to the Ritz-Galerkin problem. Show the basic error estimate (I1) below.
(c) Using the inequalities (I1)–(I4) below, show the error estimate:

$$\|u - u_h\|_{L^2(\Omega)} \le Ch^{k+1}|u|_{H^{k+1}(\Omega)}.$$

(I1) *Ritz-Galerkin basic error estimate*:

$$\|u - u_h\|_E \le \|u - v_h\|_E, \text{ for all } v_h \in V_h.$$

(I2) *Cea's Lemma*:

$$c\|u - u_h\|_{H^1(\Omega)} \le \|u - u_h\|_E \le C\|u - u_h\|_{H^1(\Omega)}.$$

(I3) *Interpolation error estimate*: Let $I_{\tau_h} u$ be the interpolant of u on the triangulation τ_h. If P_k (polynomials of degree up to k) are used inside each element

$$\|u - I_{\tau_h} u\|_{H^1(\Omega)} \le Ch^k |u|_{H^{k+1}(\Omega)}.$$

(I4) *Duality argument*:

$$\|u - u_h\|_{L^2(\Omega)} \le Ch\|u - u_h\|_{H^1(\Omega)}.$$

Note: Here c and C denote a generic positive constants. You do not need to keep track of such constants when deriving your error, also the particular definitions of the norms and seminorms are not needed in this problem.

Problem 5. Assume the non-linear BVP

$$\begin{cases} y'' = f(t, y, y'), \text{ for } t \in [0,1], \\ y(0) = \alpha \\ y(1) = \beta, \end{cases}$$

has a unique solution. Here $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function. Let $t_i = ih$, $i = 0, \ldots, n+1$, $h = 1/(n+1)$. Recall the finite difference approximations

$$y'(t_i) = \frac{1}{2h}(y(t_{i+1}) - y(t_{i-1})) + \mathcal{O}(h^2)$$
$$y''(t_i) = \frac{1}{h^2}(y(t_{i+1}) - 2y(t_i) + y(t_{i-1})) + \mathcal{O}(h^2).$$

(a) Let $y_i \approx y(t_i)$. Use the finite difference method to write the approximation to the non-linear BVP in the form

$$\mathbf{F}(\mathbf{y}) = \mathbf{0},$$

with $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{F}(\mathbf{y}) = (F_1(\mathbf{y}), \ldots, F_n(\mathbf{y}))^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$. Please be particularly careful with the equations corresponding to the end of the interval.

(b) Recall Newton's method for solving $\mathbf{F}(\mathbf{y}) = \mathbf{0}$:

Let $\mathbf{y}^{(0)}$ **be an initial guess**
for $k = 1, \ldots$
$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} - (D\mathbf{F}[\mathbf{y}^{(k)}])^{-1}\mathbf{F}(\mathbf{y}^{(k)})$
end

Find the Jacobian $D\mathbf{F}[\mathbf{y}^{(k)}] = (\nabla F_1(\mathbf{y}^{(k)}), \ldots, \nabla F_n(\mathbf{y}^{(k)}))^T \in \mathbb{R}^{n\times n}$ of $\mathbf{F}$.

(c) How does the computational cost of one Newton's method step compares to solving a linear BVP on the same grid?

(d) What can be used as initial guess $\mathbf{y}^{(0)}$?

Problem 6. The goal of this problem is to derive the update for the step size in the adaptive Runge-Kutta-Fehlberg method of orders 2 and 3.

Assume you are given two methods for solving the IVP $y' = f(t, y)$; $y(a) = \alpha$, for $t \in [a, b]$, based on

$$y(t_{i+1}) = y(t_i) + h\phi(t_i, y(t_i), h) + \mathcal{O}(h^2) \quad \text{(M1)}$$
$$y(t_{i+1}) = y(t_i) + h\widetilde{\phi}(t_i, y(t_i), h) + \mathcal{O}(h^3). \quad \text{(M2)}$$

Note that (M1) is order 1 and (M2) is order 2. Call y_i and $\widetilde{y}_i$ the approximations obtained by applying (M1) and (M2) respectively. Assuming that both methods are exact at t_i (e.g. $y_i = \widetilde{y}_i = y(t_i)$)

- the LTE for (M1) at $t_i + h$ is $\tau_{i+1}(h) = y_{i+1} - y(t_{i+1}) + \mathcal{O}(h^3)$
- the LTE for (M2) at $t_i + h$ is $\widetilde{\tau}_{i+1}(h) = \widetilde{y}_{i+1} - y(t_{i+1}) + \mathcal{O}(h^4)$

where LTE stands for Local "Truncation Error".

(a) Show that

$$\tau_{i+1}(h) = y_{i+1} - \widetilde{y}_{i+1} + \mathcal{O}(h^3).$$

(b) Assume that $\tau_{i+1}(h) = Kh^2$, where K is a constant independent of h. Let $q > 0$ be some factor that we will use to adjust the step size h. Show that

$$\tau_{i+1}(qh) = q^2\tau_{i+1}(h).$$

(c) Assume y_{i+1} and $\widetilde{y}_{i+1}$ have been calculated with step size h. Combining the previous questions we have $\tau_{i+1}(qh) \approx q^2(y_{i+1} - \widetilde{y}_{i+1})$. Given a prescribed tolerance $\epsilon > 0$, find a condition on q for which

$$|\tau_{i+1}(qh)| < \epsilon.$$